MISCELLANEOUS EXAMPLE

Ex.1 If α, β are the roots of $x^2 - p(x + 1) - c = 0$ then $\frac{\alpha^2+2\alpha+1}{\alpha^2+2\alpha+c} + \frac{\beta^2+2\beta+1}{\beta^2+2\beta+c}$ is equal to

(1) 0 (2) 1 (3) 2 (4) None of these

Sol. Here the equation is $x^2 - p(x + 1) - c = 0$

$\therefore \quad \alpha + \beta = p, \qquad \alpha\beta = -(p + c)$

$\Rightarrow \quad (\alpha + 1)(\beta + 1) = 1 - c$

Now given expression

$$= \frac{(\alpha+1)^2}{(\alpha+1)^2-(1-c)} + \frac{(\beta+1)^2}{(\beta+1)^2-(1-c)},$$ Putting value of $1 - c = (\alpha + 1)(\beta + 1)$

$$= \frac{\alpha+1}{\alpha-\beta} + \frac{\beta+1}{\beta-\alpha}$$

$$= \frac{\alpha+1-\beta-1}{\alpha-\beta} = 1$$

Hence the answer is (2)

Ex.2 The value of the expression $\frac{\sin x \cos 3x}{\sin 3x \cos x}$ cannot lie between-

(1) $-\frac{1}{3}$ and $\frac{1}{3}$ (2) $-\frac{1}{3}$ and 3 (3) -3 and $\frac{1}{3}$ (4) $\frac{1}{3}$ and 3

Sol. Here let $y = \frac{\sin x \cos 3x}{\sin 3x \cos x} = \frac{\sin x \cos 3x}{\cos x \sin 3x}$

$$= \frac{\tan x}{\tan 3x},$$ Putting $\tan x = t$.

$$y = \frac{t}{\left(\frac{3t-t^3}{1-3t^2}\right)} = \frac{t(1-3t^2)}{t(3-t^2)},$$ since $t \neq 0$ else y will be interminate

$$\therefore\ y = \frac{1-3t^2}{3-t^2} \Rightarrow t^2 = \frac{3y-1}{y-3}$$

Since $t^2 > 0 \Rightarrow \frac{3y-1}{y-3} > 0$

$$\Rightarrow \frac{(3y-1)(y-3)}{(y-3)^2} > 0.$$

Hence y can not lie between 1/3 and 3.

Ex.3 Value of x which satisfy the equation $|x-1|^{\log_3 x^2 - 2\log_x 9}$ = (x – 1)^7 is-

(1) 3^3 (2) 3^4 (3) 3^5 (4) None of these

Sol. Here x $\neq$ 1, x > 1 is necessory and L.H.S. is positive therefore R.H.S. will be compare the power $\log_3 x^2 - 2\log_x 9 = 7$

$\Rightarrow 2\log_3 x - 4\log_x 3 = 7,\ \log_3 x = y \quad \therefore \quad \log_x 3 = \frac{1}{y}$

$\Rightarrow 2y - \frac{4}{y} = 7$

$\Rightarrow 2y^2 - 7y - 4 = 0$

$\Rightarrow (y - 4)(2y + 1) = 0 \quad \therefore \quad y = 4, -\frac{1}{2}$

Now $\log_3 x = 4 \Rightarrow x = 3^4$.

Note : $y = -\frac{1}{2} \Rightarrow \log_3 x = -\frac{1}{2} \Rightarrow x = \frac{1}{\sqrt{3}}$, x > 1.

Ex.4 Which of the following equation has non real solution

(1) $2\cos^2\left(\frac{x}{2}\right)\sin^2 x = x^2 + \frac{1}{x^2}, 0 \leq x \leq \frac{\pi}{2}$ (2) $x - \frac{2}{x-1} = 1 - \frac{2}{x-1}$

(3) $e^{\cos x} - e^{-\cos x} = 4$ (4) All of these

Sol. (1) Here L.H.S. $\leq$ 2, when R.H.S. $\geq$ 2

When R.H.S. = 2 if x = 1, but L.H.S. < 2, for x = 1

hence real solution is not.

(2) If x $\neq$ 1, both the side $\frac{2}{x-1}$ is after cancelled we get x = 1 then x $\neq$ 1 is opposite. Hence no solution.

(3) Let $e^{\cos x} = y \therefore y - \frac{1}{y} = 4$

$\Rightarrow y^2 - 4y - 1 = 0 \Rightarrow y = \frac{4 \pm \sqrt{16+4}}{2} = 2 \pm \sqrt{5}$

$\therefore$ y > 0, as $e^{\cos x} > 0$, $2 - \sqrt{5}$ is respected.

Hence if $y = 2 + \sqrt{5}$

$\Rightarrow e^{\cos x} = 2 + \sqrt{5} \Rightarrow \cos x = \log_e (2 + \sqrt{5})$

But $2 + \sqrt{5} > e \Rightarrow \log_e (2 + \sqrt{5}) > \log e = 1$

$\therefore$ cos x > 1 is not true hence no solution.

Hence all there question have not any real solution.

Ex.5 If a, b, c are in G.P. then which of the following equations have equal roots -

(1) $(b - c) x^2 + (c - a)x + (a - b) = 0$

(2) $a(b - c) x^2 + b(c - a)x + c(a - b) = 0$

(3) $(a^2 + b^2) x^2 - 2b(a + c)x + (b^2 + c^2) = 0$

(4) None of these

Sol. Here From (1)

$(c - a)^2 - 4(b - c)(a - b) = 0$

$\Rightarrow (c + a - 2b)^2 = 0$

$\Rightarrow$ a, b, c in A.P.

(2) gives $[b(c - a)]^2 - 4a(b - c) c(a - b) = 0$

$\Rightarrow (ab + bc - 2ac)^2 = 0$

$\Rightarrow$ a, b, c in H. P .

(3) gives $4b^2(a + c)^2 - 4(a^2 + b^2)(b^2 + c^2) = 0$

$\Rightarrow -4(b^2 - ac)^2 = 0$

$\therefore$ a, b, c in G.P. Ans.[3]

Note :- Take a, b, c as 1, 2, 4 and find the equation by numerical substitution.

Ex.6 The number of real roots of the equation $e^{\sin x} - e^{-\sin x} - 4 = 0$ is

(1) 2 (2) 1 (3) infinite (4) None

Sol. Let $e^{\sin x} = y$ then given equation reduces to

$$y - \frac{1}{y} - 4 = 0$$

$\Rightarrow y^2 - 4y - 1 = 0$

$$\Rightarrow y = \frac{4 \pm \sqrt{20}}{2} = 2 \pm \sqrt{5}$$

$= 4.23 - 0.23$

But $y = e^{\sin x}$ is never negative. So

$y = e^{\sin x} = 4.23$

$\Rightarrow \sin x = \log 4.23 > 1.$

which is not possible. Hence the equation has no real root.

Ex.7 If the roots of the equation $(\cos p - 1) x^2 + (\cos p) x + \sin p = 0$ are real, then

(1) $p \in (0, 2\pi)$ (2) $p \in (-\pi, 0)$ (3) $p \in (-\pi/2, \pi/2)$ (4) $p \in (0, \pi)$

Sol. Roots are real

$\Rightarrow B^2 - 4AC \geq 0$

$\Rightarrow \cos^2 p - 4(\cos p - 1)\sin p \geq 0$

$\Rightarrow \cos^2 p + 4(1 - \cos p)\sin p \geq 0$

Which is possible only when $\sin p \geq 0$

i.e., when $p \in (0, \pi)$.

Ex.8 If x be real then $\frac{(x-a)(x-b)}{x-c}$ will take all real values when :

(1) $a < b < c$ (2) $a > b > c$ (3) $a < c < b$ (4) always

Sol. Let $\frac{(x-a)(x-b)}{x-c} = y$

$\Rightarrow$ $x^2 - (a + b + y)x + (ab + cy) = 0$

Now $x \in \mathbf{R}$ $\Rightarrow$ $B^2 - 4AC \geq 0$

$\Rightarrow$ $(a + b + y)^2 - 4(ab + cy) \geq 0$

$\Rightarrow$ $y^2 + y(2a + 2b - 4c) + (a - b)^2 \geq 0$

This is possible only when

$4(a + b - 2c)^2 - 4(a - b)^2 < 0$

$\Rightarrow$ $(a - c)(b - c) < 0$

$\Rightarrow$ $(c - a)(c - b) < 0$

$\Rightarrow$ c lies between a and b.

Ex.9 If $0 \leq x \leq \pi$, then the solution of the equation $16^{\sin^2 x} + 16^{\cos^2 x} = 10$ is given by x equal to -

(1) $\frac{\pi}{6}, \frac{\pi}{3}$ (2) $\frac{\pi}{3}, \frac{\pi}{2}$ (3) $\frac{\pi}{6}, \frac{\pi}{2}$ (4) None of these

Sol. $16^{\sin^2 x} = y$, then $16^{\cos^2 x} = 16^{1 - \sin^2 x} = \frac{16}{y}$

Hence $y + \frac{16}{y} = 10$ $\Rightarrow$ $y^2 - 10y + 16 = 0$ or $y = 2, 8$

Now $16^{\sin^2 x} = 2$ $\Rightarrow$ $2^{4\sin^2 x} = (2)^1$ $\Rightarrow$ $4\sin^2 x = 1$

$\therefore$ $\sin x = \pm\frac{1}{2}$ $\Rightarrow$ $x = \frac{\pi}{6}$

and $16^{\sin^2 x} = 8$ $\Rightarrow$ $2^{4\sin^2 x} = 2^3$ $\Rightarrow$ $\sin x = \pm\frac{\sqrt{3}}{2}$

$\Rightarrow$ $x = \frac{\pi}{3}$.

Ex.10 If the roots of the eqation $x^3 + bx^2 + cx = 1 = 0$ form an increasing G.P., then

(1) $b + c = 0$

(2) $b \in (-\infty, -3)$

(3) one of the roots is 1.

(4) one root is smaller then one and one root is more than 1.

Sol. Let the roots of the equation $x^3 + bx^2 + cx - 1 = 0$ be $\frac{\alpha}{r}$, α, αr where $\alpha > 0$ and $r > 1$. Then

$\frac{\alpha}{r} + \alpha + \alpha r = -b$ .... (1)

$\frac{\alpha}{r} \cdot \alpha + r, \alpha r + \frac{\alpha}{r} \cdot \alpha r = c$ .... (2)

and $\left(\frac{\alpha}{r}\right)(\alpha)(\alpha r) = 1$ .... (3)

From (3) we get $\alpha^3 = 1$ or $\alpha = 1$

From (1) we get $\frac{1}{r} + 1 + r = -b$ .... (4)

$\Rightarrow \left(\frac{1}{\sqrt{r}} - \sqrt{r}\right)^2 + 3 = -b$

$\Rightarrow -b - 3 > 0$

or $b < -3 = b \in (-\infty, -3)$

Also, from (2), $1/r + r + 1 = c$ .... (5)

From (4) and (5) $-b = c$ or $b + c = 0$

As $r > 1$, $\alpha/r = 1/r < 1$ and $\alpha r = r > 1$.

Ex.11 Let a, b, c be real numbers $a \neq 0$. If α is a root of $a^2x^2 + bx + c = 0$, β is root of $a^2x^2 - bx - c = 0$ and $0 < \alpha < \beta$; then the equation$a^2x^2 + 2bx + 2c = 0$ has a root γ which always satisfies.

(1) $\gamma = \frac{\alpha+\beta}{2}$ (2) $\gamma = \alpha + \frac{\beta}{2}$ (3) $\gamma = \alpha$ (4) $\alpha < \beta < \gamma$

Sol. We have $a^2\alpha^2 + b\alpha + c = 0$ .... (1)

$a^2\beta^2 + b\beta + c = 0$ .... (2)

Let $f(x) = a^2x^2 + 2bx + 2c$, then

$f(\alpha) = a^2\alpha^2 + 2b\alpha + 2c = a^2\alpha^2 + 2(b\alpha + c)$

$= -a^2\alpha^2$ (negative) [using (1)

$f(\beta) = a^2\beta^2 + 2(b\beta + c)$

$= 3a^2\beta^2$ (positive) [using (2)]

$\because$ $f(\alpha)$, $f(\beta)$ are of opposite sign so a root say γ of $f(x) = 0$ lies between α and β.

Ex.12 If $c \neq 0$ and the equation $\frac{p}{2x} = \frac{a}{x+c} + \frac{b}{x-c}$ has two equal roots, then p can be

(1) $(\sqrt{a} \pm \sqrt{b})^2$ (2) $(\sqrt{a} \pm \sqrt{b})$ (3) $a + b$ (4) $b + a$

Sol. We can write the given equation as $\frac{p}{2x} = \frac{(a+b)x + c(b-a)}{x^2 - c^2}$

or $p\ (x^2 - c^2) = 2\ (a + b)\ x^2 - 2c\ (a - b)\ x$

or $(2a + 2b - p)\ x^2 - 2c\ (a - b)\ x + pc^2 = 0$

For this equation to have equal roots

$c^2\ (a - b)^2 - pc^2\ (2a + 2b - p) = 0$

$\Rightarrow$ $(a - b)^2 - 2p\ (a + b) + p^2 = 0$ $[\because\ c^2 \neq 0]$

$\Rightarrow$ $[p - (a + b)]^2 = (a + b)^2 - (a - b)^2 = 4ab$

$\Rightarrow$ $p - (a + b) = \pm 2\sqrt{ab}$

$\Rightarrow$ $p = a + b \pm 2\sqrt{ab} = (\sqrt{a} \pm \sqrt{b})^2$

Hence the correct answer is (1).

Ex.13 If both roots of the equation $x^2 - 6ax + 2 - 2a + 9a^2 = 0$ exceed 3, then $a > \frac{11}{9}$.

(1) True (2) False (3) Not possible (4) None of these

Sol. The equadratic equation $f(x) = x^2 - 6ax + 2 - 2a + 9a^2 = 0$ ..... (1)

will have real roots if $D = 36a^2 - 4\ (2 - 2a + 9a^2) \geq 0$.

$\Rightarrow$ $-8\ (1 - a) \geq 0$ or $a \geq 1$ ..... (2)

The roots of (1) will exceed 3 if $-\frac{b}{2a} = -\frac{-6a}{2} = 3a > 3$ or $a > 1$ ..... (3)

and $f(3) = 9 - 18a + 2 - 2a + 9a^2 > 0$.

$\Rightarrow$ $9a^2 - 20a + 11 > 0$ $\Rightarrow$ $(9a - 11)\ (a - 1) > 0$

$\Rightarrow$ $\left(a - \frac{11}{9}\right)(a - 1) > 0 = a < 1$ or $a > \frac{11}{9}$. ..... (4)

Thus (2), (3) and (4) will hold simultaneously if $a > \frac{11}{9}$.

Ex.14 If f(x) be a quadratic expression which is positive for all real values of x and $g(x) = f(x) + f'(x) + f''(x)$; then for any real x

(1) $g(x) < 0$ (2) $g(x) > 0$ (3) $g(x) = 0$ (4) $g(x) \geq 0$

Sol. Let $f(x) = ax^2 + bx + c$, then

$g(x) = (ax^2 + bx + c)\ (2ax + b) + 2a$

$= ax^2 + (b + 2a)x + (c + b + 2a)$

$\because$ $f(x) > 0$ $\Rightarrow$ $b^2 - 4ac < 0$ and $a > 0$ ...... (1)

Now for g(x),

$B^2 - 4AC = (b + 2a)^2 - 4a(c + b + 2a)$

$= (b^2 - 4ac) - 4a^2 < 0$ [By (1)]

$\Rightarrow$ $g(x) > 0\ \forall\ x \in \mathbf{R}$

Ex.15 If $P(x) = ax^2 + bx + c$ and $Q(x) = ax^2 + dx - c$ where $ac \neq 0$, then minimum number of real roots of $P(x)\ Q(x) = 0$ is

(1) 1 (2) 2 (3) 3 (4) 4

Sol. For $P(x) = 0$, $B^2 - 4AC = b^2 - 4ac = p_1$ (say)

for $Q(x) = 0$, $B^2 - 4AC = d^2 + 4ac = p_2$ (say)

Now since $ac \neq 0$, so $ac < 0$ or $ac > 0$

When $ac < 0$ then $p_1 > 0$

$\Rightarrow$ Roots of $P(x) = 0$ real (p_2 may be positive or negative)

$\Rightarrow$ Roots of $Q(x) = 0$ real (p_1 may be positive or negative)

$\therefore$ Atleast two roots of $P(x)\ Q(x) = 0$ are real.

Ex.16 The number of real solutions of $\sin(e^x) = 5^x + 5^{-x}$ is ....

Sol. We have $5^x + 5^{-x} = (5^{x/2} - 5^{-x/2})^2 + 2 \geq 2$

If $\sin(e^x) = 5^x + 5^{-x}$ has a solution, we will get $\sin(e^x) \geq 2$, which is not possible for any real x, as $|\sin\theta| \leq 1$ for all $\theta \in \mathbf{R}$.

Ex.17 If $0 < x < 1000$ and $\left[\frac{x}{2}\right] + \left[\frac{x}{3}\right] + \left[\frac{x}{5}\right] = \frac{31}{30}x$, where [x] is the greatest integer less than or equal to x, then number of possible values of x is

(1) 34 (2) 32 (3) 33 (4) None of these

Sol. $$\left[\frac{x}{2}\right] + \left[\frac{x}{3}\right] + \left[\frac{x}{5}\right] = \frac{31}{30}x = \frac{x}{2} + \frac{x}{3} + \frac{x}{5}$$

$\Rightarrow$ $\frac{x}{2}, \frac{x}{3}, \frac{x}{5}$ are all the integers, as L.H.S. is integer.

So, x = multiple of the L.C.M. of 2, 3, 5

$\therefore$ $x = 30 \times 1, 30 \times 2, 30 \times 3, ..., 30 \times 33$

$\therefore$ x has 33 solutions.

Hence (3) is correct answer.

Ex.18 If $a \in R$, and the equation $(a - 2)(x - [x])^2 + 2(x - [x]) + a^2 = 0$ (where [x] denotes the greatest integer $\leq x$) has no integeral solution and has exactly one solution in (2, 3), then a lies in the interval.

(1) (–1, 2) (2) (0, 1) (3) (–1, 0) (4) (2, 3)

Sol. Let $y = x - [x]$.

Then equation $(a - 2)(x - [x])^2 + 2(x - [x]) + a^2 = 0$. ..... (1)

can be written as

$$f(y) = (a - 2)y^2 + 2y + a^2 = 0. \qquad ..... (2)$$

As x cannot be an integer,

$y = x - [x] \neq 0$

Thus, $a \neq 0$

When $2 < x < 3$, $[x] = 2$.

$\Rightarrow$ $0 < x - [x] < 1$ i.e. $0 < y < 1$.

Since (1) has exactly one solution in the interval (2, 3),

(2) has exactly one solution in the interval (0, 1).

This is possible if $f(0)\, f(1) < 0$

for otherwise the equation (2) has either no or two solutions in (0, 1)

$\Rightarrow$ $a^2 \{a - 2 + 2 + a^2\} < 0$

$\Rightarrow$ $a(a + 1) < 0$ $[\because a^2 > 0]$

$\Rightarrow$ $-1 < a < 0$ or $a \in (-1, 0)$

Hence (3) is correct answer.

Ex.19 If $a > 0$, $a \neq 1$, then the equation $2\log_x a + \log_{ax} a + 3\log_{a^2} a = 0$ has

(1) exactly one real root

(2) two real roots

(3) no real roots

(4) infinite number of real roots

Sol. The equation $2\log_x a + \log_{ax} a + 3\log_{a^2} a = 0$ can be written as

$$\frac{2\log a}{\log x} + \frac{2\log a}{\log(ax)} + \frac{3\log a}{\log(a^2x)} = 0 \qquad \text{....... (1)}$$

As $a > 0$ and $a \neq 1$, $\log a \neq 0$, (1) can be written as

$$\frac{2}{y} + \frac{1}{b+y} + \frac{3}{2b+y} = 0$$

Where $b = \log a$ and $y = \log x$.

$\Rightarrow$ $2(b + y)(2b + y) + y(2b + y) + 3y(b + y) = 0$

$\Rightarrow$ $4b^2 + 11by + 6y^2 = 0$

$$\Rightarrow y = \frac{-11b \pm \sqrt{121b^2 - 96b^2}}{12} = -\frac{4b}{3}, -\frac{b}{2}$$

$$\Rightarrow \log x = -\frac{4}{3}\log a \text{ or } -\frac{1}{2}\log a$$

$\Rightarrow$ $x = a^{-4/3}, a^{-1/2}$

Hence (2) is correct answer.

Ex.20 If the root of $x^2 - ax + b = 0$ are real and differ by a quantity which is less than C (C > 0). Prove that b lies between $\left(\frac{a^2 - c^2}{4}\right)$ and $\frac{a^2}{4}$.

Sol. As roots of $x^2 - ax + b = 0$ are Real and different

$$a^2 - 4b > 0$$

$$b < \frac{a^2}{4} \qquad \ldots\ldots. \text{ (i)}$$

Let the root are a and b given

$$\alpha - \beta < c$$

$$\sqrt{(\alpha+\beta)^2 - 4\alpha\beta} < c$$

$$\sqrt{a^2 - 4b} < c$$

$$a^2 - 4b < c^2 \qquad (C > 0,\ a^2 - 4b > 0)$$

Ex.21 The set of values of p for which the roots of the equation $3x^2 + 2x + p(p-1) = 0$ are of opposite sign, is

(1) $(-\infty, 0)$ (2) $(0, 1)$ (3) $(1, \infty)$ (4) $(0, \infty)$

Sol. Since the roots of the given equation are of opposite sign, product of the roots < 0

$\Rightarrow \frac{p(p-1)}{3} < 0$

$\Rightarrow p(p-1) < 0$

$\Rightarrow p \in (0, 1)$

Hence (2) is correct answer.

Ex.22 If $a \in R^-$ and $a \neq -2$, then the equation $x^2 + a|x| + 1 = 0$

(1) can not have any real root

(2) must have exactly two real roots

(3) must have either exactly two real roots or no real roots

(4) must have either four real roots or no real roots

Sol. $x^2 + a|x| + 1 = 0$

$\Rightarrow |x|^2 + a|x| + 1 = 0$

$$|x| = \frac{-a \pm \sqrt{a^2-4}}{2}$$

Since $a < 0$ and $\neq -2$

$\Rightarrow \frac{-a+\sqrt{a^2-4}}{2}$ and $\frac{-a-\sqrt{a^2-4}}{2}$ both are positive.

Thus there are four roots if $a < 2$, else no real root.

Hence (4) is correct answer.

Ex.23 If the roots of the equation $\left(1-q+\frac{p^2}{2}\right)$ $x^2 + p(1+q)x + q(q-1) + \frac{p^2}{2} = 0$ are equal then find p^2

(1) $4q$ (2) $-4q$ (3) $2q$ (4) $-2q$

Sol. As roots are equal b2 -4ac = 0

$\Rightarrow$ $b^2 = 4ac$

$\Rightarrow$ $p^2(1+q)^2 = 4\left(1-q+\frac{p^2}{2}\right)\left\{q(q-1)+\frac{p^2}{2}\right\}$

$\Rightarrow$ $p^2(1+q)^2 = \{4(1-q)+2p^2)\}\left\{q(q-1)+\frac{p^2}{2}\right\}$

$\Rightarrow$ $p^2(1+q)^2 = [-4q(1-q)^2 + 2p^2q(q-1) + 2p^2(1-q) + p^4]$

$\Rightarrow$ $p^2(1+q)^2 = -4q(1-q)^2 + p^2[2q^2 - 2q + 2 - 2q + p^2]$

$\Rightarrow$ $p^2[(1+q)^2 - 2q^2 + 4q - 2 - p^2] = -4q(1-q)^2$

$\Rightarrow$ $p^2[1 + 2q + q^2 - 2q^2 + 4q - 2 - p^2] = -4q(1-q)^2$

$\Rightarrow$ $p^2[-q^2 + 6q - 1 - p^2] = -4q(1-q)^2$

$\Rightarrow$ $p^2[-(1-q)^2 + 4q - p^2] = -4q(1-q)^2$

$\Rightarrow$ $-p^2(1-q)^2 + p^2(4q-p^2) = -4q(1-q)^2$

$\Rightarrow$ $(1-q^2)(-p^2+4q) + p^2(4q-p^2) = 0$

$\Rightarrow$ $(-p^2+4q)\{p^2+(1-q)^2\} = 0$

$\Rightarrow$ As $p^2 + (1-q)^2 \neq 0$

$-p^2 + 4q = 0$

$\Rightarrow$ $p^2 = 4q$ Hence proved

Ex.24 If $x^2 - (a-3)x + a = 0$ has at least one positive root, then

(1) $a \in (-\infty, 0) \cup [7, 9]$ (2) $a \in (-\infty, 0) \cup [7, \infty]$

(3) $a \in (-\infty, 0] \cup [9, \infty)$ (4) None of these

Sol. $x^2 - (a-3)x + a = 0$

$D = (a-3)^2 - 4a = a^2 - 10a + 9$

$D \geq 0$

$\Rightarrow$ $(a-9)(a-1)(a-1) \geq 0$

$\Rightarrow$ $a \in (-\infty, 1] \cup [9, \infty)$

Case I : When both roots are positive

$D \geq 0$ $\Rightarrow$ $(a-9)(a-1) \geq 0$

$\Rightarrow$ $D \geq 0,\ a > 3,\ a > 0$ $\Rightarrow$ $a \in [9, \infty)$ ......(1)

Case 2 : When exactly one root is positive

$\Rightarrow$ $a \leq 0$

Thus, from (1) and (2), $a \in (-\infty, 0] \cup [9, \infty)$.

Hence (3) is correct answer.

Ex.25 If roots of $x^2 - (a - 3)x + a = 0$ are such that both of them is greater than 2, then

(1) $a \in [7, 9]$ (2) $a\ [7, \infty)$ (3) $a \in [9, 10)$ (4) $a \in [7, 9)$

Sol. $x^2 - (a - 3)x + a = 0$

$\Rightarrow$ $D = (a - 3)^2 - 4a = a^2 - 10a + 9 = (a - 1)(a - 9)$

Case 1 : When both roots are greater than 2.

$$D \geq 0,\ f(2) > 0,\ -\frac{B}{2A} > 2$$

$$\Rightarrow (a - 1)(a - 9) \geq 0;\ 4 - (a - 3)2 + a > 0;\ \frac{a-3}{2} > 2$$

$\Rightarrow$ $a \in (-\infty, 1] \cup [9, \infty);\ a < 10;\ a > 7$

$a \in [9, 10)$

Hence (3) is correct answer.

Ex.26 If the qiadratic equation $ax^2 + bx + 6 = 0$ does not have real roots and $b \in R^+$, then

(1) $a > \max.\left\{\frac{b^2}{24}, b-6\right\}$ (2) $a < \max.\left\{\frac{b^2}{24}, b-6\right\}$

(3) $a > \min.\left\{\frac{b^2}{24}, b-6\right\}$ (4) $a < \min.\left\{\frac{b^2}{24}, b-6\right\}$

Sol. $ax^2 + bx + 6$, roots are not real

$\Rightarrow$ $D < 0$ $\Rightarrow$ $b^2 - 24a < 0$

$\Rightarrow$ $a > \frac{b^2}{24}$ i.e a is +ve ...... (1)

Also $f(-1) > 0$ $\Rightarrow$ $a - b + 6 > 0$

$\Rightarrow$ $a > b + 6$ ...... (2)

$\Rightarrow$ $a > \max.\left\{\frac{b^2}{24}, b-6\right\}$; using (1) and (2).

Hence (1) is correct answer.

Ex.27 α, β are the roots of the equation $P(x^2 - x) + x + 5 = 0$. If p_I, P_2 are the two value of p for which α, β are connected by $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{4}{5}$, then the value of $\frac{P_1}{P_2^2} + \frac{P_2}{P_1^2} =$

(1) 4040 (2) 4048 (3) 4840 (4) 4804

Sol. $px^2 - Px + x + 5 = 0 \Rightarrow px^2 - x(P - 1) + 5 = 0$

α and β are the roots of the equation $\Rightarrow \alpha + \beta = \frac{P-1}{P}$ and $\alpha\beta = \frac{5}{P}$ .... (i)

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{4}{5} \text{ (given)} \Rightarrow \frac{\alpha^2+\beta^2}{\alpha\beta} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta} = \frac{4}{5}$$

i.e. $\frac{(\alpha+\beta)^2}{\alpha\beta} = \frac{4}{5} + 2 = \frac{14}{5}$

$\therefore \frac{(P-1)^2}{p^2} \cdot \frac{P}{5} = \frac{14}{5}$ using (1)

i.e. $P^2 - 16P + 1 = 0$

The roots of the equation are P_1, P_2

$\therefore P_1 + P_2 = 16, \quad P_1P_2 = 1$

Now, $\frac{P_1}{P_2^2} + \frac{P_2}{P_1^2} = \frac{P_1^3 + P_2^3}{(P_1P_2)^2}$

$$= \frac{(P_1+P_2)^3 - 3P_1P_2(P_1+P_2)}{(P_1P_2)^2} = \frac{(16)^3 - 3 \times 1 \times 16}{1^2}$$

$= 16\,[(16)^2 - 3] = 16 \times 153 = 4048$ **Ans.**

Ex.28 Consider the equation $x^2 + 2x - n = 0$, where $n \in \mathbb{N}$ and $n \in [5, 1000]$. Total number of different values of 'n' so that the given equation has integral roots, is

(1) 4 (2) 8 (3) 3 (4) 6

Sol. $x^2 + 2x - n = 0 \Rightarrow (x + 1)^2 = n + 1$

$\Rightarrow x = -1 \pm \sqrt{n+1}$

Thus n + 1 should be a perfect square.

Since $n \in [5, 100] \Rightarrow n + 1 \in [6, 100]$

Number of perfect square between 1 and 100 is 10. Thus n can take 10 – 2 i.e. 8 different values.

Hence (2) is correct answer.

Ex.29 If $\log_2 (ax^2 + x + a) \geq 1 \ \forall \ x \in R$, then exhaustive set of value of 'a' is

(1) $\left(0, 1+\frac{\sqrt{5}}{2}\right)$ (2) $\left(1-\frac{\sqrt{5}}{2}, 1+\frac{\sqrt{5}}{2}\right)$ (3) $\left(0, 1-\frac{\sqrt{5}}{2}\right)$ (4) $\left(1+\frac{\sqrt{5}}{2}, \infty\right)$

Sol. $\log_2 (ax^2 + x + a) \geq 1 \ \forall \ x \in R.$

$\Rightarrow ax^2 + x + a \geq 2 \ \forall \ x \in R.$

$\Rightarrow ax^2 + x + (a - 2) \geq 0 \ \forall \ x \in R.$

$\Rightarrow$ $a > 0,\ 1 - 4a\,(a - 2) \leq 0.$

$\Rightarrow$ $4a^2 - 8a - 1 \geq 0.$

$\Rightarrow$ $a > 0,\ a \in \left(-\infty, 1 - \frac{\sqrt{5}}{2}\right) \cup \left(1 + \frac{\sqrt{5}}{2}, \infty\right)$

$\Rightarrow$ $a \in \left(1 + \frac{\sqrt{5}}{2}, \infty\right)$

Hence (4) is correct answer.

Ex.30 If $x^2 - x + a - 3 < 0$ for atleast one negative value of x, then complete set of values of 'a' is

(1) $(-\infty, 4)$ (2) $(-\infty, 2)$ (3) $(-\infty, 3)$ (4) $(-\infty, 1)$

Sol. The equation $x^2 - x + a - 3 = 0$ must have at least one negative root.

For real roots, $D \geq 0$

$\Rightarrow$ $1 - 4\,(a - 3) \geq 0$

$\Rightarrow$ $a \leq \frac{13}{4}$ $\Rightarrow$ $a \in \left(-\infty, \frac{13}{4}\right]$ ...... (1)

Both root will be non-negative if :

$D \geq 0,\ a - 3 \geq 0,\ 1 \geq 0$

$\Rightarrow$ $a \leq \frac{13}{4},\ a \geq 3$ $\Rightarrow$ $a \in \left[3, \frac{13}{4}\right]$ ...... (2)

Thus equation will at least one negative root if

$a \in \left(-\infty, \frac{13}{4}\right] \cup \left[3, \frac{13}{4}\right]$ (Using (1) and (2))

$\Rightarrow$ $a \in (-\infty, 3)$

Hence (3) is correct answer.

Ex.31 If $a, b, c \in R$, $a \neq 0$ and the quadratic equation $ax^2 + bx + c = 0$ has no real roots, then $(a + b + c)\,c$

(1) > 0 (2) < 0 (3) $= 0$ (4) None of these

Sol. Let $f(x) = ax^2 + bx + c$.

Since equation $ax^2 + bx + c = 0$ i.e. equation $f(x) = 0$ has no real root, therefore f(x) will have same sign for all real values of x.

$\Rightarrow$ f(0) and f(1) will have same sign.

$\Rightarrow$ $f(1).f(0) > 0$

$\Rightarrow$ $(a + b + c)\,c > 0.$

Ex.32 If $a_1, a_2, a_3, \ldots, a_n$ $(n \geq 2)$ are real and $(n - 1)a_1^2 - 2na_2 > 0$, than at least two roots of the equation $x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_n = 0$ are

(1) Imaginary (2) real (3) zero (4) None of these

Sol. Let $\alpha_1, \alpha_2, \alpha_3 \ldots\ldots, \alpha_n$ are the roots of the given equation

then $\Sigma\alpha_1 = \alpha_1, \alpha_2, \alpha_3 \ldots\ldots, \alpha_n = -a_1$

and $\Sigma\alpha_1\alpha_2 = \alpha_1\alpha_2, \alpha_1\alpha_3 \ldots\ldots, \alpha_{n-1}\alpha_n = a_2$

Now $(n - 1)a_1^2 - 2na_2 = (n - 1)(\Sigma\alpha_1)^2 - 2n\Sigma\alpha_1\alpha_2$

$$= \sum_{1\leq i<j\leq n}\sum (\alpha_i - \alpha_j)^2$$

But given that $(n - 1)a_1^2 - 2na_2 < 0$

$$\therefore \sum_{1\leq i<j\leq n}\sum (\alpha_i - \alpha_j)^2 < 0$$

Which is true only when at least two roots are imaginary.

Hence the correct answer is (1)

Ex.33 If $x_1, x_2, x_3, \ldots\ldots x_n$ are the roots of $x^n = ax + b = 0$ then the value of $(x_1 - x_2)(x_1 - x_3)(x_1 - x_4) \ldots\ldots\ldots (x_1 - x_n)$ is equal to

(1) $nx_1 + b$ (2) $n(x_1)^{n-1}$ (3) $n(x_1)^{n-1} + a$ (4) $n(x_1)^{n-1} + b$

Sol. $x^n + ax + b = (x - x_1)(x - x_2) \ldots\ldots\ldots (x - x_n)$

$$\Rightarrow (x - x_2)(x - x_3) \ldots\ldots\ldots (x - x_n) = \frac{x^n + ax + b}{(x - x_1)}$$

$$\Rightarrow (x_1 - x_2)(x_1 - x_3) \ldots\ldots\ldots (x_1 - x_n) = \lim_{x\to x_1} \frac{x^n + ax + b}{x - x_1}$$

$$= nx_1^{n-1} + a.$$

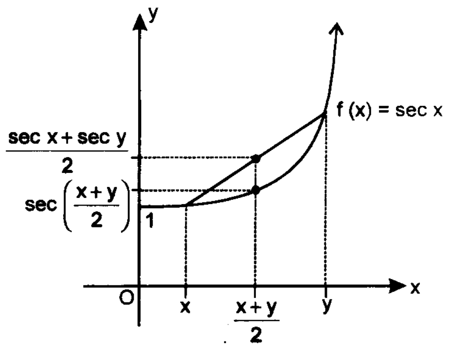
Hence (3) is correct answer.

Ex.34 The minimum positive values of x and y for which $x + y = \frac{\pi}{2}$ and $\sec x + \sec y = 2\sqrt{2}$ has solution is

(1) $x = \frac{\pi}{4}, y = -\frac{\pi}{4}$ (2) $x = y = \frac{\pi}{2}$

(3) $x = y = \frac{\pi}{4}$ (4) None of these

Sol.



As from figure : $\frac{\sec x + \sec y}{2} \geq \sec\left(\frac{x+y}{2}\right)$

$\Rightarrow$ $\frac{\sec x + \sec y}{2} \geq \sec\left(\frac{\pi}{4}\right)$

$\Rightarrow$ $\sec x + \sec y \geq 2\sqrt{2}$

Equality holds only when $\sec x = \sec y = \sqrt{2}$

$\Rightarrow$ $x = y = \frac{\pi}{4}$

Hence (3) is correct answer.

Ex.35 The values of 'a' for which $x^2 + ax + \sin^{-1}(x^2 - 4x + 5) + \cos^{-1}(x^2 - 4x + 5) = 0$, has alteast one solution, is

(1) -2 (2) $-2 + \pi$ (3) $-\frac{\pi}{4}$ (4) $-2 - \frac{\pi}{4}$

Sol. Since, $\sin^{-1}\theta$ is defined only when

$-1 \leq \theta \leq 1$

$\Rightarrow$ $-1 \leq x^2 - 4x + 5 \leq 1$

$\Rightarrow$ $x^2 - 4x + 4 \leq 0$ and $x^2 - 4x + 6 \geq 0$.

$\Rightarrow$ $(x - 2)^2 \leq 0$ and $(x - 2)^2 + 2 \geq 0$.

$\Rightarrow$ $x = 2$ is only solution.

Putting $x = 2$, we get

$$4 + 2a + \frac{\pi}{2} = 0$$

$\Rightarrow$ $a = -2 - \frac{\pi}{4}$.

Hence (4) is corret answer.

Ex.36 If $\frac{a^3(x-b)(x-c)(x-d)}{(a-b)(a-c)(a-d)}+\frac{b^3(x-c)(x-d)(x-a)}{(b-c)(b-d)(b-a)}+\frac{c^3(x-d)(x-a)(x-b)}{(c-d)(c-a)(c-b)}+\frac{d^3(x-a)(x-b)(x-c)}{(d-a)(d-b)(d-c)} = x^3$,

then x has :

(1) no solution (2) one real and two imaginary roots

(3) three real roots (4) infinitly many roots

Sol. Let

$$f(x)=\frac{a^3(x-b)(x-c)(x-d)}{(a-b)(a-c)(a-d)}+\frac{b^3(x-c)(x-d)(x-a)}{(b-c)(b-d)(b-a)}+\frac{c^3(x-d)(x-a)(x-b)}{(c-d)(c-a)(c-b)}+\frac{d^3(x-a)(x-b)(x-c)}{(d-a)(d-b)(d-c)} = x^3,$$

Where $f(a) = a^3 - a^3 = 0,$ $f(b) = 0,$

$f(c) = 0,$ $f(d) = 0,$

$\therefore$ x = a, b, c are roots of f(x) = 0.

But (x) = 0 is a polynomial of degree 3 in x. So it cann't have more than three roots. If it has more than three roots, it is identity in x. So f(x) = 0 has infinity many solutions.

Hence (4) is correct answer.

Ex.37 The equation $x^2 . 2^{|x-3|+4} + 2^{x-1} = x^2 . 2^{x+1} + 2^{|x-3|+2}$ has

(1) no solution (2) two solutions

(3) $x = \pm\frac{1}{2}$ and $x \geq 3$ (4) $x \geq 3$

Sol. Here; $x^2 . 2^{|x-3|+4} + 2^{x-1} = x^2 . 2^{x+1} + 2^{|x-3|+2}$

If; $x - 3 \geq 0$

$\Rightarrow$ $x^2 . 2^{x-3+4} + 2^{x-1} = x^2 . 2^{x+1} + 2^{x-3+2}$

$\Rightarrow$ $x^2 . 2^{x+1} + 2^{x-1} = x^2 . 2^{x+1} + 2^{x-1}$

Which is always true for $x \geq 3$.

Again if; $x - 3 < 0$

$x^2 . 2^{-(x-3)+4} + 2^{x-1} = x^2 . 2^{x+1} + 2^{-(x-3)+2}$

$\Rightarrow$ $x^2 . 2^{7-x} + 2^{x-1} = x^2 . 2^{x+1} + 2^{5-x}$

$\Rightarrow$ $x^2 . 2^{7-x} - 2^{5-x} = x^2 . 2^{x+1} - 2^{x-1}$

$\Rightarrow$ $2^{5-x}(x^2 . 2^2 - 1) = 2^{x-1}(x^2 . 2^2 - 1)$

$\Rightarrow$ $(2^{5-x} - 2^{x-1})(4x^2 - 1) = 0$

$\Rightarrow$ $x^2 = \frac{1}{4}$ and $5 - x = x - 1$

$\Rightarrow$ $x = \pm\frac{1}{2}$ and $x = 3$.

$\therefore$ Solution set is $x = \pm\frac{1}{2}$ and $x \geq 3$.

Hence (3) is correct answer.

Ex.38 For any real value of $\theta \neq \pi$, the value of the expression y = $\frac{\cos^2\theta - 1}{\cos^2\theta + \cos\theta}$ is

(1) $-1 \leq y \leq 2$ (2) $y < 0$ and $y > 2$

(3) $-1 \leq y \leq 1$ (4) $y \geq 1$

Sol. Here; $y = \frac{\cos^2\theta - 1}{\cos^2\theta + \cos\theta}$

$\Rightarrow$ $(y - 1)\cos^2\theta + y\cos\theta + 1 = 0$

$\Rightarrow$ $\cos\theta = -1$ and $\cos\theta = \frac{1}{1-y}$

$\therefore$ $-1 < \frac{1}{1-y} < 1$

$\Rightarrow$ $\frac{1}{1-y} + 1 > 0$ and $\frac{1}{1-y} - 1 < 0$

$\Rightarrow$ $\frac{2-y}{1-y} > 0$ and $\frac{y}{1-y} < 0$

$\Rightarrow$ $y < 0$ or $y > 2$

Hence (2) is correct answer.

Ex.39 Number of integers, which satisfy the inequality $\frac{(16)^{1/x}}{(2^{x+3})} > 1$, is equal to

(1) infinite (2) 0 (3) 1 (4) 4

Sol. Here, $\frac{(16)^{1/x}}{(2^{x+3})} > 1$

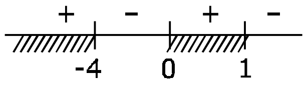
$\Rightarrow$ $\frac{2^{4/x}}{2^{x+3}} > 1$ or $2^{\frac{4}{x}-x-3} > 1$

i.e. $2^{\frac{4}{3}-x-3} > 2^o$

$\Rightarrow$ $\frac{4}{x} - x - 3 > 0$

$\Rightarrow$ $-\frac{(x^2+3x-4)}{x} > 0$ or $\frac{-(x+4)(x-1)}{x} > 0$

using number line rule :



$\Rightarrow$ $x \in (-\infty, -4) \cup (0, 1)$

Hence (1) is correct answer.

Ex.40 The equation $(x - n)^m + (x - n^2)^m + (x - n^3)^m + \ldots\ldots + (x - n^m)^m = 0$, (m is odd positive integer), has

(1) all real roots

(2) one real and (n – 1) imaginary roots

(3) one real and (m – 1) imaginary roots

(4) no real root

Sol. Here, $f(x) = (x - n)^m + (x - n^2)^m + (x - n^3)^m + \ldots\ldots + (x - n^m)^m$

$f'(x) = m(x - n)^{m-1} + m(x - n^2)^{m-1} + \ldots\ldots + m(x - n^m)^{m-1} = 0$

(as, m is odd $\Rightarrow$ (m – 1) is even)

$\therefore$ $f'(x) > 0$, as m is odd positive integer.

So, $f'(x) = 0$ has no real root.

$\Rightarrow$ $f(x) = 0$ has one real and (m – 1) imaginary roots.

Hence (3) is correct answer.