MATHEMATICS

SECTION-I

Straight Objective Type

This section contains 6 multiple choice questions. Each question has 4 choices (A), (B), (C) and (D), out of which **ONLY ONE** is correct.

Q.1 Consider the two curves

$C_1 : y^2 = 4x$

$C_2 : x^2 + y^2 - 6x + 1 = 0$

then,

(A) C_1 and C_2 touch each other only at one point

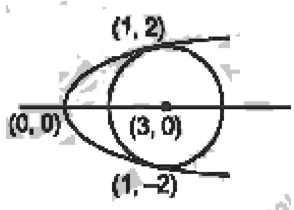
(B) C_1 and C_2 touch each other exactly at two points

(C) C_1 and C_2 intersect (but do not touch) at exactly two points

(D) C_1 and C_2 neither intersect nor touch each other

Sol. (B)

For the point of intersection of the two given curves $C_1 : y^2 = 4x$ and $C_2 : x^2 + y^2 - 6x + 1 = 0$



We have

$x^2 + 4x - 6x + 1 = 0$

$\Rightarrow x^2 - 2x + 1 = 0$

$\Rightarrow (x - 1)^2 = 0$

$\Rightarrow x = 1, 1$ equal real roots

$\Rightarrow y = 2, -2$

Thus the given curves touch each other at exactly two points (1, 2) and (1, –2)

Q.2 If $0 < x < 1$, then $\sqrt{1+x^2}\ [\{x\cos(\cot^{-1} x) + \sin(\cot^{-1} x)\}^2 - 1]^{1/2} =$

(A) $\frac{x}{\sqrt{1+x^2}}$ (B) x (C) $x\sqrt{1+x^2}$ (D) $\sqrt{1+x^2}$

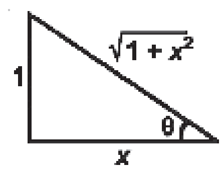
Sol. (C)

We have $0 < x < 1$,

Let $\cot^{-1} x = \theta$

$\Rightarrow \cot\theta = x$

$\Rightarrow \sin\theta = \frac{1}{\sqrt{1+x^2}} = \sin(\cot^{-1} x)$



and $\cos\theta = \frac{x}{\sqrt{1+x^2}} = \cos(\cot^{-1} x)$

Now $\sqrt{1+x^2}\ [\{x\cos(\cot^{-1} x) + \sin(\cot^{-1} x)\}^2 - 1]^{1/2}$

$$= \sqrt{1+x^2}\ \left[\left\{x\frac{x}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+x^2}}\right\}^2 - 1\right]^{\frac{1}{2}}$$

$$= \sqrt{1+x^2}\left[\left(\frac{1+x^2}{\sqrt{1+x^2}}\right)^2 - 1\right]^{\frac{1}{2}}$$

$$= \sqrt{1+x^2}\,[1 + x^2 - 1]^{1/2} = x\sqrt{1+x^2}$$

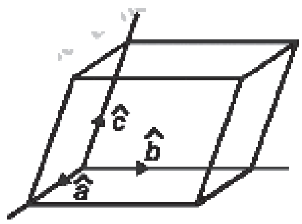
Q.3 The edges of a parallelopiped are of unit length and are parallel to non-coplanar unit vectors $\hat{a}, \hat{b}, \hat{c}$ such that $\hat{a}\cdot\hat{b} = \hat{b}\cdot\hat{c} = \hat{c}\cdot\hat{a} = \frac{1}{2}$. Then, the volume of the parallelopiped is

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{2\sqrt{2}}$ (C) $\frac{\sqrt{3}}{2}$ (D) $\frac{1}{\sqrt{3}}$

Sol. **(A)**

The volume of the parallelopiped with coterminus edges as $\hat{a}, \hat{b}, \hat{c}$ is given by $[\hat{a}\,\hat{b}\,\hat{c}] = \hat{a}\cdot(\hat{b}\times\hat{c})$

Now $[\hat{a}\,\hat{b}\,\hat{c}]^2 = \begin{vmatrix} \hat{a}.\hat{a} & \hat{a}.\hat{b} & \hat{a}.\hat{c} \\ \hat{b}.\hat{a} & \hat{b}.\hat{b} & \hat{b}.\hat{c} \\ \hat{c}.\hat{a} & \hat{c}.\hat{b} & \hat{c}.\hat{c} \end{vmatrix} = \begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{vmatrix} = \frac{1}{2}$



$\Rightarrow [\hat{a}\,\hat{b}\,\hat{c}]^2 = \frac{1}{2}$

Thus the required volume of the parallelopiped = $\frac{1}{\sqrt{2}}$ cubic units.

Q.4 Let a and b be non-zero real numbers. Then, the equation
$(ax^2 + by^2 + c)(x^2 - 5xy + 6y^2) = 0$ represents
(A) Four straight lines, when c = 0 and a, b are of same sign
(B) Two straight lines and a circle, when a = b, and c is of sign opposite to that of a
(C) Two straight lines and a hyperbola, when a and b are of the same sign and c is of sign opposite to that of a
(D) A circle and an ellipse, when a and b are of the same sign and c is of sign opposite to that of a

Sol. **(B)**

Let a and b be non-zero real numbers

The equation $(ax^2 + by^2 + c)(x^2 - 5xy + 6y^2) = 0$ implies either

$x^2 - 5xy + 6y^2 = 0$

$\Rightarrow (x - 2y)(x - 3y) = 0$

$\Rightarrow x = 2y$ and $x = 3y$

$\Rightarrow x^2 - 5xy + 6y^2 = 0$ represents two straight lines passing through origin.

or $ax^2 + by^2 + c = 0$

when c = 0 and a and b are of same signs then

$ax^2 + by^2 + c = 0 \Rightarrow x = 0$ and $y = 0$

which is a point specified as the origin.
when a = b and c is of sign opposite of that of a, $ax^2 + by^2 + c = 0$ represents a circle
Hence the given equation $(ax^2 + by^2 + c)(x^2 - 5xy + 6y^2) = 0$ may represent two straight lines and a circle.

Q.5 Let $g(x) = \frac{(x-1)^n}{\log\cos^m(x-1)}$; $0 < x < 2$, m and n are integers, $m \neq 0$, $n > 0$, and let p be the left hand derivative of $|x-1|$ at $x = 1$. If $\lim_{x \to 1+} g(x) = p$, then

(A) n = 1, m = 1 (B) n = 1, m = −1 (C) n = 2, m = 2 (D) n > 2, m = n

Sol. **(C)**

We have, $g(x) = \frac{(x-1)^n}{\log\cos^m(x-1)}$, $0 < x < 2$, $m \neq 0$, n are integers

and $|x - 1| = \begin{cases} x-1; & x \geq 1 \\ 1-x; & x < 1 \end{cases}$

The left hand derivative of $|x - 1|$ at $x = 1 = p = -1$

Also $\lim_{x \to 1+} g(x) = p = -1$

$$\Rightarrow \lim_{h \to 0} \frac{(1+h-1)^n}{\log\cos^m(1+h-1)} = -1$$

$$\Rightarrow \lim_{h \to 0} \frac{h^n}{\log\cos^m h} = -1$$

$$\Rightarrow \lim_{h \to 0} \frac{h^n}{m\log\cos h} = -1$$

$$\Rightarrow \lim_{h \to 0} \frac{n.h^{n-1}}{m\frac{1}{\cos h}(-\sin h)} = -1$$

$$\Rightarrow \lim_{h \to 0} \left(-\frac{n}{m}\right) \cdot \frac{h^{n-2}}{\left(\frac{\tan h}{h}\right)} = -1$$

$$\Rightarrow \left(\frac{n}{m}\right) \lim_{h \to 0} \frac{h^{n-2}}{\left(\frac{\tan h}{h}\right)} = 1$$

$\Rightarrow n = 2$ and $\frac{n}{m} = 1$

$\Rightarrow m = n = 2$

Q.6 The total number of local maxima and local minima and local minima of the function

$$f(x) = \begin{cases} (2+x)^3 & ;-3 < x \le -1 \\ x^{\frac{2}{3}} & ;-1 < x < 2 \end{cases} \text{ is}$$

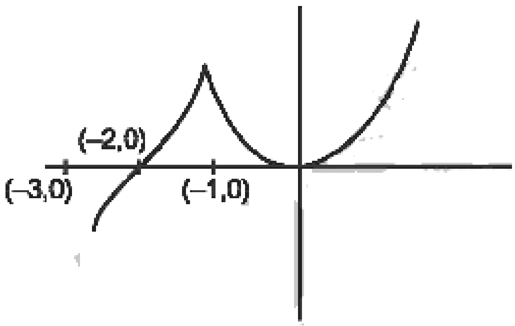
(A) 0 (B) 1 (C) 2 (D) 3

Sol. **(C)**

We have,

$$f(x) = \begin{cases} (2+x)^3 & ;-3 < x \le -1 \\ x^{\frac{2}{3}} & ;-1 < x < 2 \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} 3(x+2)^2 & ;-3 < x \le -1 \\ \frac{2}{3}x^{-\frac{1}{3}} & ;-1 < x < 2 \end{cases}$$



Clearly f'(x) changes its sign at x = – 1 from + ve to – ve and so if (x) has local maxima at x = –1

also f'(0) does not exist but f'(0-) < 0 and f'(0+) > 0, it can only be inferred that f(x) has a possibility of a minima at x = 0

Hence the given function has one local maxima at x = – 1 and one local minima at x = 0

SECTION-II

Multiple Correct Answers Type

This section contains 4 multiple correct answer(s) type questions. Each question has 4 choices (A), (B), (C) and (D), out of which **ONE OR MORE** is/are correct.

Q.7 A straight line through the vertex P of a triangle PQR intersects the side QR at the point S and the circumcircle of the triangle PQR at the point T. If S is not the centre of the circumcircle, then

(A) $\frac{1}{PS} + \frac{1}{ST} < \frac{2}{\sqrt{QS \times SR}}$ (B) $\frac{1}{PS} + \frac{1}{ST} > \frac{2}{\sqrt{QS \times SR}}$

(C) $\frac{1}{PS} + \frac{1}{ST} < \frac{4}{QR}$ (D) $\frac{1}{PS} + \frac{1}{ST} > \frac{4}{QR}$

Sol. **(B, D)**

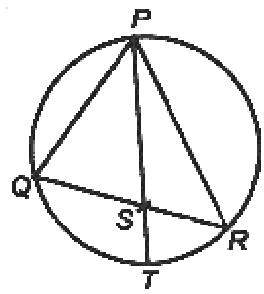
Let a straight line through the vertex P of a given ΔPQR intersects the side QR at the point S and the circumcircle of ΔPQR at the point T.

points P,Q,R,T are concyclic, hence

PS.ST= QS.SR

Now $\frac{PS+ST}{2} \geq \sqrt{PS.ST}$

and $\frac{1}{PS} + \frac{1}{ST} \geq \frac{2}{\sqrt{PS.ST}} = \frac{2}{\sqrt{QS.SR}}$



Also $\frac{SQ+SR}{2} \geq \sqrt{SQ.SR}$

$\Rightarrow \frac{QR}{2} \geq \sqrt{SQ.SR}$

$\Rightarrow \frac{1}{\sqrt{SQ.SR}} \geq \frac{2}{QR} \Rightarrow \frac{2}{\sqrt{QS.SR}} \geq \frac{4}{QR}$

Hence $\frac{1}{PS} + \frac{1}{ST} \geq \frac{2}{\sqrt{QS.SR}} \geq \frac{4}{QR}$

Q.8 Let P (x_1, y_1) and Q (x_2, y_2), $y_1 < 0$, $y_2 < 0$, be the end points of the latus rectum of the ellipse $x^2 + 4y^2 = 4$

The equations of parabolas with latus rectum PQ are

(A) $x^2 + 2\sqrt{3}y = 3 + \sqrt{3}$ (B) $x^2 - 2\sqrt{3}y = 3 + \sqrt{3}$

(C) $x^2 + 2\sqrt{3}y = 3 - \sqrt{3}$ (D) $x^2 - 2\sqrt{3}y = 3 - \sqrt{3}$

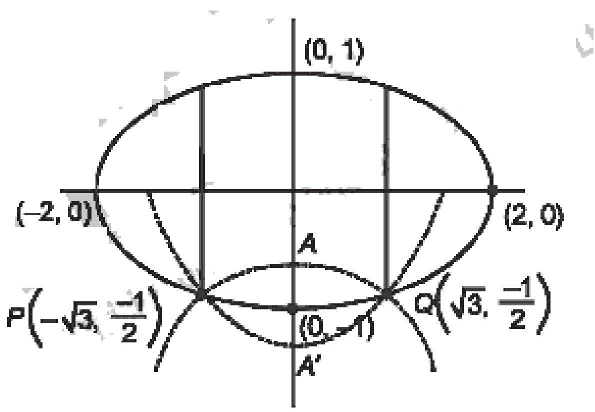
Sol. **(B, C)**

The equation $x^2 + 4y^2 = 4$ represents an ellipse with 2 and 1 as semi-major and semi-minor axes and eccentricity $\frac{\sqrt{3}}{2}$. Thus the ends of latera recta are

$\left(\sqrt{3},\frac{1}{2}\right)$, $\left(\sqrt{3},-\frac{1}{2}\right)$, $\left(-\sqrt{3},\frac{1}{2}\right)$ and $\left(-\sqrt{3},-\frac{1}{2}\right)$

According to the question, P $\left(-\sqrt{3},-\frac{1}{2}\right)$ and Q $\left(\sqrt{3},-\frac{1}{2}\right)$

$PQ = 2\sqrt{3}$



Thus the coordinates of the vertex of the parabolas are A $\left(0, \frac{-1+\sqrt{3}}{2}\right)$ and A' $\left(0, \frac{-1-\sqrt{3}}{2}\right)$ and corresponding equation are

$(x - 0)^2 = 4\frac{\sqrt{3}}{2}\left(y + \frac{1-\sqrt{3}}{2}\right)$

and $(x - 0)^2 = 4\frac{\sqrt{3}}{2}\left(y - \frac{-1-\sqrt{3}}{2}\right)$

i.e $x^2 + 2\sqrt{3}\,y = 3 - \sqrt{3}$

and $x^2 - 2\sqrt{3}\,y = 3 + \sqrt{3}$

Q.9 Let $S_n = \sum_{k=1}^{n} \frac{n}{n^2+kn+k^2}$ and $T_n = \sum_{k=0}^{n-1} \frac{n}{n^2+kn+k^2}$, for n = 1,2,3 .... then

(A) $S_n < \frac{\pi}{3\sqrt{3}}$ (B) $S_n > \frac{\pi}{3\sqrt{3}}$ (C) $T_n < \frac{\pi}{3\sqrt{3}}$ (D) $T_n > \frac{\pi}{3\sqrt{3}}$

Sol. (A, D)

We have

$$S_n = \sum_{k=0}^{n-1} \frac{n}{n^2+kn+k^2} = \sum_{k=0}^{n-1} \frac{1}{n}\left(\frac{1}{1+\frac{k}{n}+\frac{k^2}{n^2}}\right) < \lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{1}{n}\left(\frac{1}{1+\frac{k}{n}+\left(\frac{k}{n}\right)^2}\right)$$

$$= \int_0^1 \frac{1}{1+x+x^2}dx = \left[\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right)\right]_0^1$$

$$= \frac{2}{\sqrt{3}} . \left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{\pi}{3\sqrt{3}}$$

i.e. $S_n < \frac{\pi}{3\sqrt{3}}$

Similarly $T_n > \frac{\pi}{3\sqrt{3}}$

Q.10 Let f(x) be a non- constant twice differentiable function defined on $(-\infty, \infty)$ such that $f(x) = f(1- x)$ and $f'\left(\frac{1}{4}\right) = 0$, Then

(A) f'(x) vanishes at least twice on [0,1] (B) $f'\left(\frac{1}{2}\right) = 0$

(C) $\int_{-\frac{1}{2}}^{\frac{1}{2}} f\left(x+\frac{1}{2}\right)\sin x\,dx = 0$ (D) $\int_0^{\frac{1}{2}} f(t)e^{\sin \pi t}\,dt = \int_{\frac{1}{2}}^{1} f(1-t)e^{\sin \pi t}dt$

Sol. (All)

We have

$f(x) = f(1 - x)$ differentiating it w.r.t. x, we get

$f'(x) = -f'(1 - x)$

Put $x = \frac{1}{2}$

$\Rightarrow 2f'\left(\frac{1}{2}\right) = 0$

$\Rightarrow f'\left(\frac{1}{2}\right) = 0$

Since $f'\left(\frac{1}{2}\right) = 0$ and $f'\left(\frac{1}{4}\right) = 0$

$\Rightarrow$ f'' (x) = 0 at two points in (0,1)

Now $\int_{-\frac{1}{2}}^{\frac{1}{2}} f\left(x+\frac{1}{2}\right) \sin x \, dx = 0$

as $f\left(x+\frac{1}{2}\right) \sin x$ is an odd function which is clear from the following explanation

$g(x) = f\left(x+\frac{1}{2}\right) \sin x$, say,

$g(-x) = f\left(\frac{1}{2}-x\right) \sin(-x) = -\sin x\, f\left(1-\left(\frac{1}{2}-x\right)\right) = -\sin x\, f\left(\frac{1}{2}+x\right) = -g(x)$

Moreover $\int_{-\frac{1}{2}}^{1} f(1-t)e^{\sin(\pi t)} dt = \int_{0}^{\frac{1}{2}} f(u)e^{\sin \pi u} du$

Considering 1–t = u.

SECTION-III

Assertion - Reason Type

The section contains 4 reasoning type questions. Each question has 4 choices (A), (B), (C) and (D), out of which **ONLY ONE** is correct.

Q.11 Let f and g be real valued functions defined on interval (–1,1) such that g'' (x) is continuous, $g(0) \neq 0$, $g'(0) = 0$, $g''(0) \neq 0$, and $f(x) = g(x) \sin x$.

STATEMENT-1 ; $\lim_{x \to 0}$ [g(x)cotx – g(0) cosec x] = f''(0).

and

STATEMENT-2 ; f'(0) = g(0)

(A) STATEMENT-1 is True, STATEMENT-2 is True ; STATEMENT-2 is a correct explanation for STATEMENT-1

(B) STATEMENT-1 is True, STATEMENT-2 is True; STATEMENT -2 is NOT a correct explanation for STATEMENT-1

(C) STATEMENT-1 is True, STATEMENT-2 is False

(D) STATEMENT-1 is False, STATEMENT-2 is True

Sol. **(B)**

We have $\lim_{x \to 0} \frac{g(x)\cos x - g(0)}{\sin x}$; $\left[\frac{0}{0}\right]$ form

$= \lim_{x \to 0} \frac{g'(x)\cos x - g(x)\sin x}{\cos x}$

$= 0$

$f(x) = g(x) \sin x$

$f'(x) = g'(x) \sin x + g(x) \cos x$

$f''(x) = g''(x) \sin x + 2g'(x) \cos x - g(x) \sin x$

$\Rightarrow f''(0) = 0$

Thus $\lim_{x \to 0} [g(x) \cos x - g(0) \operatorname{cosec} x] = 0 = f''(0)$

$\Rightarrow$ Statement–1 is True.

Statement-2 $f'(x) = g'(x) \sin x + g(x) \cos x$

$\Rightarrow f'(0) = g(0)$

Statement-2 is True but not a correct explanation of Statement-1

Q.12 Consider three planes

p_1 ; $x - y + z = 1$

p_2 ; $x + y - z = -1$

p_3 ; $x - 3y + 3z = 2$

Let L_1, L_2, L_3 be the lines of intersection of the planes p_2 and p_3, p_3 and p_1, p_1 and p_2, respectively

STATEMENT-1 ; At least two of the lines L_1, L_2 and L_3 are non parallel.

and

STATEMENT-2; The three planes do not have a common point.

(A) STATEMENT-1 is True, STATEMENT-2 is True; STATEMENT-2 is a correct explanation for STATEMENT-1

(B) STATEMENT-1 is True, STATEMENT-2 is True; STATEMENT-2 is NOTa correct exoplanation for STATEMENT-1

(C) STATEMENT-1 is True, STATEMENT-2 is False

(D) STATEMENT-1 is False, STATEMENT-2 is True

Sol. **(D)**

The given equations are

$x - y + z = 1$

$x + y - z = -1$

$x - 3y + 3z = 2$

The system of equations can be put in matrix form as $Ax = B$

i.e. $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -2 \\ 0 & -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ $R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$

$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$

Which is inconsistent as $\rho(A : B) \neq \rho(A)$

$\Rightarrow$ The three planes do not have common point

Statment-2 is True

Since planes p_1, p_2, p_3 are pairwise intersecting, their lines of intersection are parallel

Statement-1, is False

Q.13 Consider the system of equations

$x - 2y + 3z = -1$

$-x + y - 2z = k$

$x - 3y + 4z = 1$

STATEMENT-1; The system of equations has no solution for $k \neq 3$

and

STATEMENT - 2 The determinant $\begin{vmatrix} 1 & 3 & -1 \\ -1 & -2 & k \\ 1 & 4 & 1 \end{vmatrix} \neq 0$, for $k \neq 3$.

(A) STATEMENT-1 is True, STATEMENT -2 is True; STATEMENT -2 is a correct explanation for STATEMENT -1

(B) STATEMENT -1 is True, STATEMENT-2 is True; STATEMENT -2 is NOT a correct explanation for STATEMENT-1

(C) STATEMENT -1 is True, STATEMENT -2 is False

(D) STATEMENT -1 is False, STATEMENT -2 is True

Sol. **(A)**

The given system of equations can be expressed as

$\begin{bmatrix} 1 & -2 & 3 \\ 1 & -3 & 4 \\ -1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ k \end{bmatrix}$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 + R_1$

$$\Rightarrow \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ k-1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -2 & 3 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ k-3 \end{bmatrix} \text{ by } R_3 \to R_3 - R_2$$

When k $\neq$ 3. the given system of equations has no solution

$\Rightarrow$ Statement 1 is true. Clearly statement -2 is also true as it is rearrangement of rows and columns of $\begin{bmatrix} 1 & -2 & 3 \\ 1 & -3 & 4 \\ -1 & 1 & -2 \end{bmatrix}$

Q.14 Consider the system of equations

ax + by = 0, cx + dy = 0, where a, b, c, d $\in$ {0, 1}

STATEMENT-1 ; The probability that the system of equations has a unique solution is $\frac{3}{8}$

and

STATEMENT-2 ; The probability that the system of equations has a solution is 1

(A) STATEMENT-1 is True, STATEMENT-2 is True; STATEMENT-2 is a correct explanation for STATEMENT-1

(B) STATEMENT-1 is True, STATEMENT-2 is True; STATEMENT-2 is NOT a correct explanation for STATEMENT-1

(C) STATEMENT-1 is True, STATEMENT-2 is False

(D) STATEMENT-1 is False, STATEMENT -2 is True

Sol. (B)

The number of all possible determinants of the form $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 2^4 = 16$

Out of which only 10 determinants given by

$$\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}$$

Vanish and remaining six determinants have non-zero value.

Hence the required probability = $\frac{6}{16} = \frac{3}{8}$

$\Rightarrow$ Statement-1 is True

Statement-2 is also True as the homogeneous equations have always a solution & Statement-2 does not explain the explanation of Statement-1.

SECTION-IV

Linked Comprehension Type

This section contains 3 paragraphs. Based upon each paragraph, 3 multiple choice questions have to be answered. Each question has 4 choices (A), (B), (C) and (D), out of which **ONLY ONE** is correct.

Paragraph for Question Nos. 15 to 17

A circle C of radius 1 is inscribed in an equilateral triangle PQR. The point of contact of C with the sides PQ, QR & RP are D, E and F respectively. The line PQ is given by the equation $\sqrt{3}x + y - 6 = 0$ and the point D is $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$. Further, it is given that the origin and the centre of C are on the same side of the line PQ

Q.15 The equation of circle C is

(A) $(x - 2\sqrt{3})^2 + (y-1)^2 = 1$

(B) $(x - 2\sqrt{3})^2 + \left(y + \frac{1}{2}\right)^2 = 1$

(C) $(x - \sqrt{3})^2 + (y + 1)^2 = 1$

(D) $(x - \sqrt{3})^2 + (y - 1)^2 = 1$

Sol. (D)

Let centre of C be (h,k)

Then,

$$\left|\frac{\sqrt{3}h + k - 6}{\sqrt{3+1}}\right| = 1$$

$\Rightarrow \sqrt{3}h + k - 6 = 2, -2$

$\Rightarrow \sqrt{3}h + k = 4$ .... (i) (Rejecting 2 because origin and centre of C are on the same side of PQ)

The point $(\sqrt{3}, 1)$ satisfies equation (i)

Therefore equation of circle C is $(x - \sqrt{3})^2 + (y - 1)^2 = 1$

Q.16 Points E and F are given by

(A) $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right), (\sqrt{3}, 0)$

(B) $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), (\sqrt{3}, 0)$

(C) $\left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right), \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

(D) $\left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right), \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

Sol. (A)

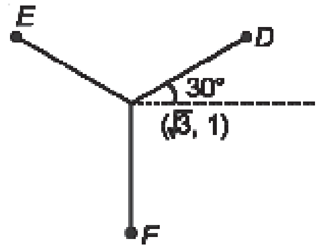
Slope of line joining centre of circle to point D is

$$\frac{\frac{3}{2} - 1}{\frac{3\sqrt{3}}{2} - \sqrt{3}} = \frac{1}{\sqrt{3}}$$ (Makes an angle 30^0 with x-axis)

$\therefore$ points E and F will make angle 150^0 and -90^0 with x-axis

$\therefore$ E and F are given by

$$\frac{x-\sqrt{3}}{\cos 150^0} = \frac{y-1}{\sin 150^0} = 1 \text{ and } \frac{x-\sqrt{3}}{\cos(-90^0)} = \frac{y-1}{\sin(-90^0)} = 1$$



$$\therefore \; E = \left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right) \text{ and } F = (\sqrt{3}, 0)$$

Q.17 Equations of the sides QR, RP are

(A) $y = \frac{2}{\sqrt{3}}x + 1,\; y = -\frac{2}{\sqrt{3}}x - 1$ (B) $y = \frac{1}{\sqrt{3}}x,\; y = 0$

(C) $y = \frac{\sqrt{3}}{2}x + 1,\; y = -\frac{\sqrt{3}}{2}x - 1$ (D) $y = \sqrt{3}x,\; y = 0$

Sol. (D)

Clearly, point E and F satisfy the equations given in option (D)

Paragraph for Question Nos. 18 to 20

Consider the functions defined implicitly by the equation $y^3 - 3y + x = 0$ on various intervals in the real line. If $x \in (-\infty, -2) \cup (2, \infty)$, the equation implicitly defines a unique real valued differentiable function $y = f(x)$. If $x \in (-2, 2)$ the equation implicitly defines a unique real valued differentiable function $y = g(x)$ satisfying $g(0) = 0$

Q.18 If $f(-10\sqrt{2}) = 2\sqrt{2}$, then $f''(-10\sqrt{2}) =$

(A) $\frac{4\sqrt{2}}{7^3 3^2}$ (B) $-\frac{4\sqrt{2}}{7^3 3^2}$ (C) $\frac{4\sqrt{2}}{7^3 3}$ (D) $-\frac{4\sqrt{2}}{7^3 3}$

Sol. (B)

At $x = -10\sqrt{2}$, $y = f(x)$

$$\therefore f''(x) = \frac{d^2y}{dx^2}$$

$$y^3 - 3y + x = 0 \Rightarrow 3y^2\frac{dy}{dx} - 3\frac{dy}{dx} + 1 = 0 \qquad \text{...........(i)}$$

$$\Rightarrow 3y^2\left(\frac{d^2y}{dx^2}\right) + 6y\left(\frac{dy}{dx}\right)^2 - 3\frac{d^2y}{dx^2} = 0 \qquad \text{........... (ii)}$$

At $x = -10\sqrt{2}$, $y = 2\sqrt{2}$

Substituting in (i)

$$3(2\sqrt{2})^2\frac{dy}{dx} - 3\frac{dy}{dx} + 1 = 0$$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{-1}{21}$

Substituting in (ii)

$3(2\sqrt{2})^2 . \dfrac{d^2y}{dx^2} + 6(2\sqrt{2}) . \left(\dfrac{-1}{21}\right)^2 - 3 . \dfrac{d^2y}{dx^2} = 0$

$\Rightarrow 21 . \dfrac{d^2y}{dx^2} = \dfrac{-12\sqrt{2}}{(21)^2}$

$\Rightarrow \dfrac{d^2y}{dx^2} = \dfrac{-12\sqrt{2}}{(21)^3} = \dfrac{-4\sqrt{2}}{7^3 3^2}$

Q.19 The area of the region bounded by the curve y = f(x), the x-axis, and lines x = and x = b, where $-\infty < a < b < -2$, is

(A) $\int_a^b \dfrac{x}{3[(f(x))^2 - 1]} dx + bf(b) - af(a)$ (B) $-\int_a^b \dfrac{x}{3[(f(x))^2 - 1]} dx + bf(b) - af(a)$

(C) $\int_a^b \dfrac{x}{3[(f(x))^2 - 1]} dx - bf(b) + af(a)$ (D) $-\int_a^b \dfrac{x}{3[(f(x))^2 - 1]} dx - bf(b) + af(a)$

Sol. (A)

Given area = $\int_a^b y dx = \int_a^b f(x) dx$

$= [f(x).x]_a^b - \int_a^b f'(x).x dx$

$= bf(b) - af(a) - \int_a^b f'(x) x dx$

$= bf(b) - af(a) + \int_a^b \dfrac{x dx}{3[(f(x))^2 - 1]}$

as $f'(x) = \dfrac{dy}{dx} = \dfrac{-1}{3(y^2 - 1)} = \dfrac{-1}{3[(f(x))^2 - 1]}$

Q.20 $\int_{-1}^{1} g'(x) dx =$

(A) 2g(−1) (B) 0 (C) −2g(1) (D) 2g(1)

Sol. (D)

Let $I = \int_{-1}^{1} g'(x) dx = [g(x)]_{-1}^{1} = g(1) - g(-1)$

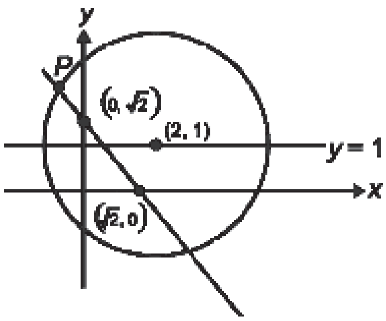
Since $y^3 - 3y + x = 0$ ....(i)
and $y = g(x)$
hence $(g(x))^3 - 3g(x) + x = 0$ by (i)
at $x = 1$, $(g(1))^3 - 3g(1) + 1 = 0$ ....(ii)
at $x = -1$, $(g(-1))^3 - 3g(-1) - 1 = 0$ ....(iii)
(ii) + (iii)
$(g(1))^3 + (g(-1))^3 - 3(g(1) + g(-1)) = 0$
$(g(1) + g(-1))\,[(g(1))^2 + (g(-1))^2 - g(1)\,g(-1) - 3] = 0$
$\Rightarrow g(1) + g(-1) = 0$
$\Rightarrow g(1) = -g(-1)$
$\Rightarrow I = g(1) - g(-1) = g(1) - (-g(1)) = 2g(1)$

Paragraph for Question Nos.21 to 23

let A,B,C be three sets of complex numbers as defined below
$A = \{Z;\ \text{Im}Z \geq 1\}$
$B = \{Z;|Z - 2 - i| = 3\}$
$C = \{Z;\ \text{Re}((1 - i)Z) = \sqrt{2}\}$

Q.21 The number of elements in the set $A \cap B \cap C$ is
(A) 0 (B) 1 (C) 2 (D) ∞

Sol. (B)
Let $Z = x + iy$
Set A corressponds to the region $y \geq 1$ ....(i)
Set B consists of points lying on the circle, centred at (2,1) and radius 3
i.e. $x^2 + y^2 - 4x - 2y = 4$ ....(ii)
Set C consists of points lying on the $x + y = \sqrt{2}$ ....(iii)



Clearly, there is only one point of intersection of the line $x + y = \sqrt{2}$, and circle $x^2 + y^2 - 4x - 2y = 4$

Q.22 Let Z be any point in $A \cap B \cap C$. Then $|Z + 1 - i|^2 + |z - 5 - i|^2$ lies between
(A) 25 and 29 (B) 30 and 34 (C) 35 and 39 (D) 40 and 44

Sol. (C)
$|Z + 1 - i|^2 + |Z - 5 - i|^2$
$= (x + 1)^2 + (y-1)^2 + (x - 5)^2 + (y-1)^2$
$= 2(x^2 + y^2 - 4x - 2y) + 28$
$= 2(4) + 28$ $(\because x^2 + y^2 - 4x - 2y = 4)$
$= 36$

Q.23 Let z be any point in $A \cap B \cap C$ and let w be any point satisfying $|w - 2 - i| < 3$. Then, $|z| - |w| + 3$ lies between

(A) –6 and 3 (B) –3 and 6 (C) – 6 and 6 (D) –3 and 9

Sol. **(D)**

$|w - (2 + i)| < 3$

$\Rightarrow \; ||w| - |2 + i|| < 3$

$\Rightarrow \; -3 + \sqrt{5} < |w| < 3 + \sqrt{5}$

$\Rightarrow \; -3 - \sqrt{5} < -|w| < 3 - \sqrt{5}$ ....(i)

Also, $|z - (2 + i)| = 3$

$\Rightarrow \; -3 + \sqrt{5} \le |z| \le 3 + \sqrt{5}$

$\therefore \; -3 < |z| - |w| + 3 < 9$