MATHEMATICS

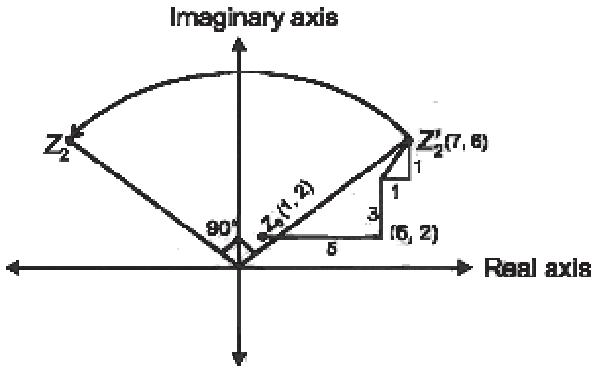
SECTION-I

Straight Objective Type

This section contains 6 multiple choice questions. Each question has 4 choices (A), (B), (C) and (D), out of which **ONLY ONE** is correct.

Q.1 A particle P starts from the point $Z_0 = 1 + 2i$, where $i = \sqrt{-1}$. It moves first horizontally away from origin by 5 units and then vertically away from origin by 3 units to reach a point Z_1. From Z_1 the particle moves $\sqrt{2}$ units in the direction of the vector $\hat{i} + \hat{j}$ and then it moves through an angle $\frac{\pi}{2}$ in anticlockwise direction on a circle with centre at origin, to reach a point Z_2. The point Z_2 is given by

(A) $6 + 7\hat{i}$ (B) $-7 + 6\hat{i}$ (C) $7 + 6\hat{i}$ (D) $-6 + 7\hat{i}$



Sol. (D)

$Z'_2 = (6 + \sqrt{2}\cos 45^0, 5 + \sqrt{2}\sin 45^0) = (7, 6) = 7 + 6i$

by rotation about (0, 0)

$\frac{Z_2}{Z'_2} = e^{i\frac{\pi}{2}} \Rightarrow Z_2 = Z'_2 \left(e^{i\frac{\pi}{2}}\right)$

$Z_2 = (7 + 6i)\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = (7 + 6i)(i) = -6 + 7i$

Q.2 Let the function $g: (-\infty, \infty) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ be given by $g(u) = 2\tan^{-1}(e^u) - \frac{\pi}{2}$. Then, g is

(A) Even and is strictly increasing in $(0, \infty)$
(B) Odd and is strictly decreasing in $(-\infty, \infty)$
(C) Odd and is strictly increasing in $(-\infty, \infty)$
(D) Neither even nor odd, but is strictly increasing in $(-\infty, \infty)$

Sol. (C)

$g(u) = 2\tan^{-1}(e^u) - \frac{\pi}{2}$ for $u \in (-\infty, \infty)$

$g(-u) = 2\tan^{-1}(e^{-u}) - \frac{\pi}{2}$

$= 2(\cot^{-1}(e^u)) - \frac{\pi}{2}$

$= 2\left(\frac{\pi}{2} - \tan^{-1}(e^u)\right) - \frac{\pi}{2}$

$= \pi - 2\tan^{-1}(e^u) - \frac{\pi}{2}$

$= \frac{\pi}{2} - 2\tan^{-1}(e^u) = -g(u)$

$g(-u) = -g(u) \Rightarrow g(u)$ is an odd function

Q.3 Consider a branch of the hyperbola $x^2 - 2y^2 - 2\sqrt{2}x - 4\sqrt{2}y - 6 = 0$ with vertex at the point A. Let B be one of the end points of its latus rectum. If C is the focus of the hyperbola nearest to the point A, then the area of the triangle ABC is

(A) $1 - \sqrt{\frac{2}{3}}$ (B) $\sqrt{\frac{3}{2}} - 1$ (C) $1 + \sqrt{\frac{2}{3}}$ (D) $\sqrt{\frac{3}{2}} + 1$

Sol. (B)

$\frac{(x-\sqrt{2})^2}{4} - \frac{(y+\sqrt{2})^2}{2} = 1$

for A(x, y)

$e = \sqrt{1+\frac{2}{4}} = \sqrt{\frac{3}{2}}$

$x - \sqrt{2} = 2 \Rightarrow x = 2 + \sqrt{2}$

For C (x, y)

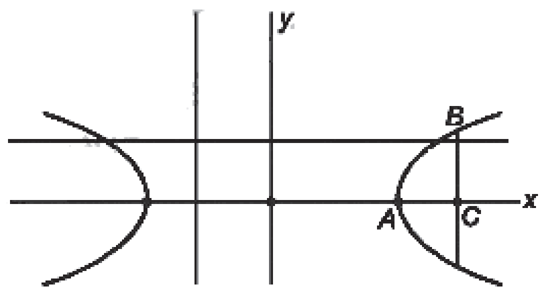
$x - \sqrt{2} = ae = \sqrt{6}$

$x = \sqrt{6} + \sqrt{2}$

$AC = \sqrt{6} + \sqrt{2} - 2 - \sqrt{2} = \sqrt{6} - 2$

$BC = \frac{b^2}{a} = \frac{2}{2} = 1$

Area $= \frac{1}{2} \times BC \times AC = \sqrt{\frac{3}{2}} - 1$



Q.4 The area of the region between the curves $y = \sqrt{\frac{1+\sin x}{\cos x}}$ and $y = \sqrt{\frac{1-\sin x}{\cos x}}$ bounded by the lines $x = 0$ and $x = \frac{\pi}{4}$ is

(A) $\int_0^{\sqrt{2}-1} \frac{t}{(1+t^2)\sqrt{1-t^2}}\,dt$ (B) $\int_0^{\sqrt{2}-1} \frac{4t}{(1+t^2)\sqrt{1-t^2}}\,dt$

(C) $\int_0^{\sqrt{2}+1} \frac{4t}{(1+t^2)\sqrt{1-t^2}}\,dt$ (D) $\int_0^{\sqrt{2}+1} \frac{t}{(1+t^2)\sqrt{1-t^2}}\,dt$

Sol. (B)

$$\int_0^{\frac{\pi}{4}} \left(\sqrt{\frac{1+\sin x}{\cos x}} - \sqrt{\frac{1-\sin x}{\cos x}}\right) dx \qquad \left(\because \frac{1+\sin x}{\cos x} > \frac{1-\sin x}{\cos x} > 0\right)$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{\frac{1+\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}{\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}} - \sqrt{\frac{1-\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}{\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}}\, dx$$

$$= \int_0^{\frac{\pi}{4}} \left(\sqrt{\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}} - \sqrt{\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}}\right) dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{2\tan\frac{x}{2}}{\sqrt{1-\tan^2\frac{x}{2}}}\, dx$$

Put $\tan\frac{x}{2} = t$, $\frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$

$$= \int_0^{\tan\frac{\pi}{8}} \frac{4t}{(1+t^2)\sqrt{1-t^2}}\,dt$$

as $\tan\frac{\pi}{8} = \sqrt{2} - 1$

So $\int_0^{\sqrt{2}-1} \frac{4t}{(1+t^2)\sqrt{1-t^2}}\,dt$

Q.5 Consider three points $P = (-\sin(\beta - \alpha), -\cos\beta)$, $Q = (\cos(\beta - \alpha), \sin\beta)$ and $R = (\cos(\beta - \alpha + \theta), \sin(\beta - \theta))$, where $0 < \alpha, \beta, \theta < \frac{\pi}{4}$. Then,

(A) P lies on the line segment RQ
(B) Q lies on the line segment PR
(C) R lies on the line segment QP
(D) P, Q, R are non-collinear

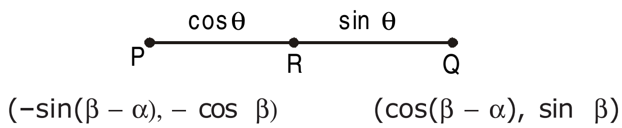
Sol. **(D)**

For collinear points

$$\begin{vmatrix} -\sin(\beta-\alpha) & -\cos\beta & 1 \\ \cos(\beta-\alpha) & \sin\beta & 1 \\ \cos(\beta-\alpha+\theta) & \sin(\beta-\theta) & 1 \end{vmatrix}$$

Clearly $\Delta \neq 0$ for any value α, β, θ hence points are non- collinear

Ailter method : (by observation)



In the case $R = \left(\frac{\cos\theta.\cos(\beta-\alpha) - \sin\theta\sin(\beta-\alpha)}{\sin\theta + \cos\theta}, \frac{\cos\theta\sin\beta - \sin\theta\cos\beta}{\sin\theta+\cos\theta}\right)$

$$R = \left(\frac{\cos(\beta-\alpha+\theta)}{\sin\theta+\cos\theta}, \frac{\sin(\beta-\theta)}{\sin\theta+\cos\theta}\right)$$

$= (\cos(\beta-\alpha+\theta), \sin(\beta-\theta))$, if $\sin\theta + \cos\theta) = 1$

Which is not possible if $0 < \theta < \frac{\pi}{4}$. Hence points are collinear.

Q.6 An experiment has 10 equally likely outcomes. Let A and B be two non-empty events of the experiment. If A consists of 4 outcomes, the number of outcomes that must have so that. A and B are independent is

(A) 2, 4 or 8
(B) 3, 6 or 9
(C) 4 or 8
(D) 5 or 10

Sol. **(D)**

$P(A) = \frac{2}{5}$

For independent events

$P(A \cap B) = P(A)\, P(B)$

$P(A \cap B) \leq \frac{2}{5}$

$\Rightarrow P(A \cap B) = \frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10}$ (Maximum 4 outcomes may be in $A \cap B$)

(i) $P(A \cap B) = \frac{1}{10}$

$\Rightarrow P(A)\,P(B) = \frac{1}{10}$

$P(B) = \frac{1}{10} \times \frac{5}{2} = \frac{1}{4}$, Not possible

(ii) $P(A \cap B) = \frac{2}{10} \Rightarrow \frac{2}{5} \times P(B) = \frac{2}{10}$

$\Rightarrow P(B) = \frac{5}{10}$, Outcomes of B = 5

(iii) $P(A \cap B) = \frac{3}{10} \Rightarrow P(A)\,P(B) = \frac{3}{10}$

$\frac{2}{5} \times P(B) = \frac{3}{10}$

$P(B) = \frac{3}{4}$, not possible

(iv) $P(A \cap B) = \frac{4}{10}$

$\Rightarrow P(A).\,P(B) = \frac{4}{10} \Rightarrow P(B) = 1$, Outcomes of B = 10

Q.7 Let two non-collinear unit vectors $\hat{a}$ and $\hat{b}$ form an acute angle. A point P moves so that at any time t the position vector $\vec{OP}$ (where O is the origin) is given by $\hat{a}\cos t + \hat{b}\sin t$. When P is farthest from origin O, let M be the length of $\vec{OP}$ and $\hat{u}$ be the unit vector along $\vec{OP}$. Then

(A) $\hat{u} = \frac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}$ and $M = \left(1+\hat{a}.\hat{b}\right)^{\frac{1}{2}}$

(B) $\hat{u} = \frac{\hat{a}-\hat{b}}{|\hat{a}-\hat{b}|}$ and $M = \left(1+\hat{a}.\hat{b}\right)^{\frac{1}{2}}$

(C) $\hat{u} = \frac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}$ and $M = \left(1+2\hat{a}.\hat{b}\right)^{\frac{1}{2}}$

(D) $\hat{u} = \frac{\hat{a}-\hat{b}}{|\hat{a}-\hat{b}|}$ and $M = \left(1+2\hat{a}.\hat{b}\right)^{\frac{1}{2}}$

Sol. (A)

$\vec{OP} = \hat{a}\cos t + \hat{b}\sin t$

$|\vec{OP}| = \sqrt{\left(\hat{a}.\hat{a}\cos^2 t + \left(\hat{b}.\hat{b}\right)\sin^2 t + 2\hat{a}.\hat{b}\sin t\cos t\right)}$

$|\vec{OP}| = \sqrt{1+2\hat{a}.\hat{b}\sin t\cos t}$

$|\vec{OP}| = \sqrt{1+\hat{a}.\hat{b}\sin 2t}$

$|\vec{OP}|_{max} = \sqrt{1+\hat{a}.\hat{b}}$ at $\sin 2t = 1 \Rightarrow t = \frac{\pi}{4}$

$\vec{OP}$ at $(t = \frac{\pi}{4}) = \frac{1}{\sqrt{2}}(\hat{a} + \hat{b})$

unit vector along $\vec{OP}$ at $\left(t = \frac{\pi}{4}\right) = \frac{\hat{a}+\hat{b}}{|\hat{a}+\hat{b}|}$

Q.8 Let $I = \int \frac{e^x}{e^{4x}+e^{2x}+1}dx$, $J = \int \frac{e^x}{e^{-4x}+e^{-2x}+1}dx$

Then, for an arbitrary constant C, the value of J–I equals

(A) $\frac{1}{2}\log\left(\frac{e^{4x}-e^{2x}+1}{e^{4x}+e^{2x}+1}\right) + C$

(B) $\frac{1}{2}\log\left(\frac{e^{2x}+e^{x}+1}{e^{2x}-e^{x}+1}\right) + C$

(C) $\frac{1}{2}\log\left(\frac{e^{2x}-e^{x}+1}{e^{2x}+e^{x}+1}\right) + C$

(D) $\frac{1}{2}\log\left(\frac{e^{4x}+e^{2x}+1}{e^{4x}-e^{2x}+1}\right) + C$

Sol. **(C)**

$$J = \int \frac{e^{3x}}{1+e^{2x}+e^{4x}}dx$$

$$J–I = \int \frac{(e^{3x}-e^{x})}{1+e^{2x}+e^{4x}}dx$$

$$= \int \frac{(u^2-1)}{1+u^2+u^4}\,du \qquad (u = e^x)$$

$$= \int \frac{\left(1-\frac{1}{u^2}\right)du}{1+\frac{1}{u^2}+u^2}$$

$$= \int \frac{\left(1-\frac{1}{u^2}\right)du}{\left(u+\frac{1}{u}\right)^2-1}$$

$$= \int \frac{dt}{t^2-1} \qquad \left(t = u+\frac{1}{u}\right)$$

$$= \frac{1}{2}\log\left|\frac{t-1}{t+1}\right| + C$$

$$= \frac{1}{2}\log\left|\frac{u^2 - u + 1}{u^2 + u + 1}\right| + C$$

$$= \left|\frac{e^{2x} - e^x + 1}{e^{2x} + e^x + 1}\right| + C$$

Q.9 Let g(x) = logf(x), where f(x) is a twice differentiable positive function on $(0, \infty)$ such that f(x + 1) = x f(x) then for N = 1, 2, 3......

$$g''\left(N + \frac{1}{2}\right) - g''\left(\frac{1}{2}\right) =$$

(A) $-4\left\{1 + \frac{1}{9} + \frac{1}{25} + ... + \frac{1}{(2N-1)^2}\right\}$

(B) $4\left\{1 + \frac{1}{9} + \frac{1}{25} + ... + \frac{1}{(2N-1)^2}\right\}$

(C) $-4\left\{1 + \frac{1}{9} + \frac{1}{25} + ... + \frac{1}{(2N+1)^2}\right\}$

(D) $4\left\{1 + \frac{1}{9} + \frac{1}{25} + ... + \frac{1}{(2N+1)^2}\right\}$

Sol. (A)

$f(x) = e^{g(x)} \Rightarrow e^{g(x+1)} = f(x + 1)$

$= xf(x) = xe^{g(x)}$

$g(x + 1) = \log x + g(x)$

i.e. $g(x + 1) - g(x) = \log x$ .....(1)

Replacing x by $x - \frac{1}{2}$

$$g\left(x + \frac{1}{2}\right) - g\left(x - \frac{1}{2}\right) = \log\left(x - \frac{1}{2}\right)$$

$$= \log(2x - 1) - \log 2$$

$$\therefore g'\left(x + \frac{1}{2}\right) - g'\left(x - \frac{1}{2}\right) = \frac{2}{2x - 1}$$

$$\therefore g''\left(x + \frac{1}{2}\right) - g''\left(x - \frac{1}{2}\right) = \frac{-4}{(2x-1)^2} \quad .....(2)$$

Substituting. x = 1, 2, 3........ N in (2) and adding

$$g''\left(N + \frac{1}{2}\right) - g''\left(\frac{1}{2}\right) = -4\left\{1 + \frac{1}{9} + \frac{1}{25} + ... + \frac{1}{(2N-1)^2}\right\}$$

SECTION-II
REASONING TYPE

This section contains 4 reasoning type questions. Each question has 4 choices (A), (B), (C) and (D), out of which **ONLY ONE** is correct.

Q.10 Suppose four distinct positive numbers a_1, a_2, a_3, a_4 are in G.P. Let $b_1 = a_1$, $b_2 = b_1 + a_2$, $b_3 = b_2 + a_3$ and $b_4 = b_3 + a_4$.

STATEMENT-1 : The numbers b_1, b_2, b_3, b_4 are neither in A.P. nor in G.P.

and

STATEMENT-2 : The numbers b_1, b_2, b_3, b_4 are in H.P.

(A) STATEMENT-1 is true, STATEMENT-2 is true; STATEMENT-2 is correct explanation for STATEMENT-1
(B) STATEMENT-1 is true, STATEMENT-2 is true; STATEMENT-2 is not a correct explanation for STATEMENT-1
(C) STATEMENT-1 is true, STATEMENT-2 is false
(D) STATEMENT-1 is false, STATEMENT-2 is true.

Sol. (C)

Let $a_1 = 1$, $a_2 = 2$, $a_3 = 4$, $a_4 = 8$

$\therefore$ $b_1 = 1$, $b_2 = 3$, $b_3 = 7$, $b_4 = 15$

Clearly b_1, b_2, b_3, b_4 are not in H.P.

Statement-2 is false

Q.11 Let a, b, c, p, q be real numbers. Suppose α, β are the roots of the equation $x^2 + 2px + q = 0$ and $\alpha, \frac{1}{\beta}$ are the roots of the equation $ax^2 + 2bx + c = 0$ where $\beta^2 \notin \{-1, 0, 1\}$

STATEMENT-1 : $(p^2 - q)(b^2 - ac) \geq 0$

and

STATEMENT-2 : $b \neq pa$ or $c \neq qa$

(A) STATEMENT-1 is true, STATEMENT-2 is true; STATEMENT-2 is correct explanation for STATEMENT-1
(B) STATEMENT-1 is true, STATEMENT-2 is true ; STATEMENT-2 is not a correct explanation for STATEMENT-1
(C) STATEMENT-1 is true, STATEMENT-2 is false
(D) STATEMENT-1 is false, STATEMENT-2 is true.

Sol. (B)

$x^2 + 2px + q = 0$

$\alpha + \beta = -2p$ .....(i)

$\alpha\beta = q$ .....(ii)

$ax^2 + 2bx + c = 0$

$\alpha + \frac{1}{\beta} = \frac{-2b}{a}$ ......(iii)

$\frac{\alpha}{\beta} = \frac{c}{a}$ ....(iv)

$$(p^2 - q)\ (b^2 - ac) = \left(\left(\frac{\alpha+\beta}{-2}\right)^2 - \alpha\beta\right) \left(\left(\frac{\alpha+\frac{1}{\beta}}{2}\right)^2 - \frac{\alpha}{\beta}\right) a^2$$

$$= \frac{(\alpha-\beta)^2}{16}\left(\alpha - \frac{1}{\beta}\right)^2 .\ a^2 \geq 0 \text{ Statement-1 is true}$$

Now $pa = -\left(\frac{\alpha+\beta}{2}\right)a = \frac{a}{2}(\alpha + \beta)$

$b = -\frac{a}{2}\left(\alpha + \frac{1}{\beta}\right)$

$pa \neq b \Rightarrow \alpha + \frac{1}{\beta} \neq \alpha + \beta \Rightarrow \beta^2 \neq 1,\ \beta \neq \{-1, 0, 1\}$, correct

Similarly

If $c \neq qa \Rightarrow a\frac{\alpha}{\beta} \neq a\alpha\beta$

$\Rightarrow \alpha\left(\beta - \frac{1}{\beta}\right) \neq 0$

$\Rightarrow \alpha \neq 0$, and $\beta - \frac{1}{\beta} \neq 0 \Rightarrow \beta \neq \{-1,0,1\}$

Statement-2 is true.

Both statement-1 and statement-2 are true, But Statement-2 do not explains statement-1

Q.12 Consider

$L_1 : 2x + 3y + p - 3 = 0$

$L_2 : 2x + 3y + p + 3 = 0$, Where p is a real number

and $C : x^2 + y^2 + 6x - 10y + 30 = 0$

STATEMENT-1 : If line L_1 is a chord of circle C, then line L_2 is not always a diameter of circle C and

STATEMENT-2 : If line L_1 is a diameter of circle C, then line L_2 is not a chord of circle C

(A) STATEMENT-1 is true, STATEMENT-2 is true; STATEMENT-2 is correct explanation for STATEMENT-1
(B) STATEMENT-1 is true, STATEMENT-2 is true; STATEMENT-2 is not a correct explanation for STATEMENT-1
(C) STATEMENT-1 is true, STATEMENT-2 is false
(D) STATEMENT-1 is false, STATEMENT-2 is true.

Sol. (C)

$(x + 3)^2 + (y - 5)^2 = 9 + 25 - 30 = 4$

$(x + 3)^2 + (y - 5)^2 = 2^2$

Centre = (3, – 5)

If L_1 is diameter, then 2(3) + 3(–5) + p – 3 = 0

p = 12

$\therefore$ L_1 is 2x + 3y + 9 = 0

L_2 is 2x + 3y + 15 = 0

Distance of centre of circle from L_2 equals

$$\left|\frac{2(3)+3(-5)+15}{\sqrt{2^2+3^2}}\right| = \frac{6}{\sqrt{13}} < 2 \text{ (radius of circle)}$$

$\therefore$ L_2 is a chord of circle C.

Statement-2 is false

Q.13 Let a solution y = y (x) of the differential equation

$x\sqrt{x^2-1}\,dy - y\sqrt{y^2-1}\,dx = 0$, satisfy $y(2) = \frac{2}{\sqrt{3}}$

STATEMENT-1 : $y(x) = \sec\left(\sec^{-1}x - \frac{\pi}{6}\right)$

and STATEMENT-2 : y(x) is given by $\frac{1}{y} = \frac{2\sqrt{3}}{x} - \sqrt{1-\frac{1}{x^2}}$

(A) STATEMENT-1 is true, STATEMENT-2 is true; STATEMENT-2 is correct explanation for STATEMENT-1

(B) STATEMENT-1 is true, STATEMENT-2 is true; STATEMENT-2 is not a correct explanation for STATEMENT-1

(C) STATEMENT-1 is true, STATEMENT-2 is false

(D) STATEMENT-1 is false, STATEMENT-2 is true.

Sol. **(C)**

$$\frac{dy}{dx} = \frac{y\sqrt{y^2-1}}{x\sqrt{x^2-1}}$$

$$\int\frac{dy}{y\sqrt{y^2-1}} = \int\frac{dy}{x\sqrt{x^2-1}}$$

$\sec^{-1} y = \sec^{-1} x + c$

given x = 2 and $y = \frac{2}{\sqrt{3}}$

Therefore $\sec^{-1}\frac{2}{\sqrt{3}} = \sec^{-1} 2 + c$

$$\frac{\pi}{6} = \frac{\pi}{3} + c$$

$$c = -\frac{\pi}{6}$$

Thus $y = \sec(\sec^{-1} x - \pi/6)$

$$= \cos\left[\cos^{-1}\frac{1}{x} - \cos^{-1}\frac{\sqrt{3}}{2}\right] = \omega$$

$$= \cos\left[\cos^{-1}\left(\frac{\sqrt{3}}{2x} + \sqrt{1-\frac{1}{x^2}}\sqrt{1-\frac{3}{4}}\right)\right]$$

$$\frac{1}{y} = \frac{\sqrt{3}}{2x} + \frac{1}{2}\sqrt{1-\frac{1}{x^2}}$$

SECTION-III

Linked Comprehension Type

The section contains 2 paragraphs. Based upon each paragraph,3 multiple choice questions have to be answered Each question has 4 choices (A), (B), (C) and (D), out of which **ONLY ONE** is correct.

Paragraph for Question Nos. 14 to 16

consider the function $f : (-\infty, \infty) \to (-\infty, \infty)$ defined by

$$f(x) = \frac{x^2 - ax + 1}{x^2 + ax + 1},\ 0 < a < 2.$$

Q.14 Which of the following is true ?

(A) $(2 + a)^2 f''(1) + (2 - a)^2 f''(-1) = 0$ (B) $(2 - a)^2 f''(1) - (2 + a)^2 f''(-1) = 0$

(C) $f'(1) f'(-1) = (2 - a)^2$ (D) $f'(1) f'(-1) = -(2 + a)^2$

Sol. (A)

$$f(x) = \frac{(x^2 + ax + 1) - 2ax}{x^2 + ax + 1}$$

$$= 1 - \frac{2ax}{x^2 + ax + 1}$$

$$f'(x) = -\left[\frac{(x^2 + ax + 1)2a - 2ax(2x + a)}{(x^2 + ax + 1)^2}\right]$$

$$= -\left[\frac{-2ax^2 + 2a}{(x^2 + ax + 1)^2}\right] = 2a\left[\frac{(x^2 - 1}{(x^2 + ax + 1)^2}\right] \qquad \text{...............(i)}$$

$$f''(x) = 2a\left[\frac{(x^2 + ax + 1)^2(2x) - 2(x^2 - 1)(x^2 + ax + 1)(2x + a)}{(x^2 + ax + 1)^4}\right]$$

$$= 2a\left[\frac{2x(x^2 + ax + 1) - 2(x^2 - 1)(2x + a)}{(x^2 + ax + 1)^3}\right]$$

$$= 4a\left[\frac{-x^3+3x+a}{\left(x^2+ax+1\right)^3}\right] \qquad ......(ii)$$

$$f''(1) = \left[\frac{4a(a+2)}{(a+2)^3}\right] = \frac{4a}{(a+2)^2}$$

$$f''(-1) = \frac{4a(a-2)}{(2-a)^3} = -\frac{4a}{(a-2)^2}$$

$\therefore$ $(2 + a)^2 f''(1) + (2 - a)^2 f''(-1) = 4a - 4a = 0$

Q.15 Which of the following is true?

(A) f(x) is decreasing on (–1,1) and has a local minimum at x = 1

(B) f(x) is increasing on (–1,1) and has a local maximum at x = 1

(C) f(x) is increasing on (–1,1) but has neither a local maximum nor a local minimum at x = 1

(D) f(x) is decreasing on (–1,1) but has neither a local maximum nor a local minimum at x = 1

Sol. (A)

When $x \in (-1, 1)$

$x^2 < 1 \Rightarrow x^2 - 1 < 0$

$\therefore$ $f'(x) < 0 \Rightarrow f(x)$ is decreasing

Also, at x = 1, $f''(1) = \frac{4a}{(a+2)^2} > 0$ (since $0 < a < 2$)

f(x) has a local minimum at x = 1

Q.16 Let $g(x) = \int_0^{e^x} \frac{f'(t)}{1+t^2}\,dt$

Which of the following is true?

(A) g'(x) is positive on $(-\infty,0)$ and negative on $(0,\infty)$

(B) g'(x) is negative on $(-\infty,0)$ and positive on $(0,\infty)$

(C) g'(x) changes sign on both $(-\infty, 0)$ and $(0,\infty)$

(D) g'(x) does not change sign on $(-\infty,\infty)$

Sol. (B)

$$g'(x) = \frac{f'\left(e^x\right)}{1+\left(e^x\right)^2}.e^x$$

$$= 2a\left[\frac{e^{2x}-1}{\left(e^{2x}+ae^x+1\right)^2}\right]\left(\frac{e^x}{1+e^{2x}}\right)$$

$g'(x) = 0$, if $e^{2x} - 1 = 0$, i.e. $x = 0$

if $x < 0$, $e^{2x} < 1 \Rightarrow g'(x) < 0$

Paragraph for Question nos. 17 to 19

Consider the lines:

$$L_1 : \frac{x+1}{3} = \frac{y+2}{1} = \frac{z+1}{2}$$

$$L_2 : \frac{x-2}{1} = \frac{y+2}{2} = \frac{z-3}{3}$$

Q.17 The unit vector perpendicular to both L_1 and L_2 is

(A) $\frac{-\hat{i}+7\hat{j}+7\hat{k}}{\sqrt{99}}$ (B) $\frac{-\hat{i}-7\hat{j}+5\hat{k}}{5\sqrt{3}}$ (C) $\frac{-\hat{i}+7\hat{j}+5\hat{k}}{5\sqrt{3}}$ (D) $\frac{7\hat{i}-7\hat{j}-\hat{k}}{\sqrt{99}}$

Sol. (B)

The equations of given lines in vector form may be written as

$L_1 : \vec{r} = (-\hat{i} - 2\hat{j} - \hat{k}) + \lambda(3\hat{i} + \hat{j} + 2\hat{k})$

and $L_2 : \vec{r} = (2\hat{i} - 2\hat{j} + 3\hat{k}) + \mu(\hat{i} + 2\hat{j} + 3\hat{k})$

$\therefore$ Vector perpendicular to both L_1 and L_2 is

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = -\hat{i} - 7\hat{j} + 5\hat{k}$$

Required unit vector $= \frac{(-\hat{i}-7\hat{j}+5\hat{k})}{\sqrt{(-1)^2+(-7)^2+(5)^2}}$

$$= \frac{1}{5\sqrt{3}}(-\hat{i}-7\hat{j}+5\hat{k})$$

Q.18 The shortest distance between L_1 and L_2 is

(A) 0 (B) $\frac{17}{\sqrt{3}}$ (C) $\frac{41}{5\sqrt{3}}$ (D) $\frac{17}{5\sqrt{3}}$

Sol. (D)

Shortest distance between L_1 and L_2 is

$$\left|\frac{\{(2-(-1))\hat{i}+(2-2)\hat{j}+(3-(-1))\hat{k}\}.(-\hat{i}-7\hat{j}+5\hat{k})}{5\sqrt{3}}\right|$$

$$= \left|\frac{(3\hat{i}+4\hat{k}).(-\hat{i}-7\hat{j}+5\hat{k})}{5\sqrt{3}}\right| = \frac{17}{5\sqrt{3}} \text{ units}$$

Q.19 The distance of the point (1, 1, 1) from the plane passing through the point (–1,–2,–1) and whose normal is perpendicular to both the lines L_1 and L_2 is

(A) $\frac{2}{\sqrt{75}}$ units (B) $\frac{7}{\sqrt{75}}$ units (C) $\frac{13}{\sqrt{75}}$ units (D) $\frac{23}{\sqrt{75}}$ units

Sol. **(C)**

The equation of the plane passing through the point (–1,–2,–1) and whose normal is perpendicular to both the given lines L_1 and L_2 may be written as

$(x + 1) + 7 (y + 2) -5 (z + 1) = 0$

i.e. $x + 7y - 5z + 10 = 0$

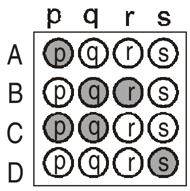
The distance of the point (1, 1, 1) from the plane = $\left|\frac{1+7-5+10}{\sqrt{1+49+25}}\right| = \frac{13}{\sqrt{75}}$ units

SECTION-IV

Matrix-Match Type

This section contains 3 questions. Each question contains statements given in two columns, which have to be matched. Statements in **Column I** are labelled as (A), (B), (C) and (D), whereas statements in **Column II** are labelled as p,q,r and s. The answers to these questions have to be appropriately bubbled as illustrated in the following example.

If the correct matches are A-p, B-q, B-r, C-p, C-q and D-s, then the correctly bubbled matrix will look like the following



Q.20 Consider the lines given by

$L_1 : x + 3y -5 = 0$

$L_2 : 3x - ky - 1 = 0$

$L_3 : 5x + 2y - 12 = 0$

Match the Statements/Expressions in **Column I** with Statements/Expressions in **Column II** and indicate your answer by darkening the appropriate bubbles in the 4 x 4 matrix given in the ORS.

Column I	**Column II**
(A) L_1, L_2, L_3 are concurrent, if	(p) $k = -9$
(B) One of L_1, L_2, L_3 is parallel to at least one of the other two, if	(q) $k = -\frac{6}{5}$
(C) L_1, L_2, L_3 form a triangle, if	(r) $k = \frac{5}{6}$
(D) L_1, L_2, L_3 do not form a triangle if	(s) $k = 5$

Sol. **A - s ; B - p, q ; C - r, D - p, q, s**

(A) Solving L_1 and L_3

$$\frac{x}{-36+10} = \frac{y}{+12-25} = \frac{1}{2-15}$$

$\therefore$ x = 2, y = 1

L_1, L_2, L_3 are concurrent, if (2, 1) lies on L_2

$\therefore$ 6 − k − 1 = 0 $\Rightarrow$ k = 5

(B) Either L_1 is parallel to L_2 or L_3 is parallel to L_2,

Then $\frac{1}{3} = \frac{3}{-k}$ or $\frac{3}{5} = \frac{-k}{2}$

$\Rightarrow$ k = − 9 or k = $\frac{-6}{5}$

(C) L_1, L_2, L_3 form a triangle, if they are not concurrent, or not parallel

$\therefore$ k $\neq$ 5, − 9 − $\frac{6}{5}$ $\Rightarrow$ k = $\frac{5}{6}$

(D) L_1, L_2, L_3 do not form a triangle, if

K = 5, − 9, − $\frac{6}{5}$

Q.21 Match the Statements/Expressions in **Column I** with Statements/Expressions in **Column II** and indicate your answer by darkening the appropriate bubbles in the 4 x 4 matrix given in the ORS.

Column I	**Column II**
(A) The minimum value of $\frac{x^2+2x+4}{x+2}$ is	(p) 0
(B) Let A and B be 3 x 3 matrices of real number, where A is symmetric, B is skew-symmetric, and (A+ B) (A − B) = (A − B) (A + B). If $(AB)^t = (-1)^k AB$ where $(AB)^t$ is the transpose of the matrix AB, then the possible values of k are	(q) 1
(C) Let a = $\log_3 \log_3 2$. An integer k satisfying $1 < 2^{(-k+3^{-a})} < 2$, must be less than	(r) 2
(D) If $\sin\theta = \cos\phi$, then the possible values of $\frac{1}{\pi}(\theta \pm \phi - \frac{\pi}{2})$ are	(s) 3

Sol. A - r ; B - q, s ; C - r, D - p, r

(A) Let y = $\frac{x^2+2x+4}{x+2}$

$\Rightarrow$ $x^2 + (2 - y)x + (4 - 2y) = 0$

$\Rightarrow$ $(2 - y)^2 - 4(4 - 2y) \geq 0$

$\Rightarrow$ $y^2 + 4y - 12 \geq 0$

$\Rightarrow$ $y \leq -6$, $y \geq 2$

$\therefore$ Minimum value of y is 2

(B) $(A + B)(A - B) = (A - B)(A + B)$

$\Rightarrow A^2 - AB + BA - B^2 = A^2 + AB - BA - B^2$

$\Rightarrow AB = BA$

$(AB)^t = (-1)^k AB \Rightarrow B^tA^t = (-1)^k AB$

$\Rightarrow - BA = (-1)^{k+1} AB$ (since $B^t = - B$, $A^t = A$)

$\Rightarrow B.A = (-1)^{k+1} AB$

$\Rightarrow (-1)^{k+1} = 1$

$\therefore$ k + 1 is even, or k is odd

(C) $1 < 2^{(-k+3^{-a})} < 2$

$\Rightarrow 0 < - k + 3^{-a} < 1$

$a = \log_3 \log_3 2 \Rightarrow 3^a = \log_3 2$

$\Rightarrow 3^{-a} = \log_2 3$ ....(1)

$\therefore k < \log_2 3 < 2$ ....(2)

and $1 + k > \log_2 3 > 1 \Rightarrow k > 0$ ....(3)

$\therefore k < 2$

by (2) and (3) $0 < k < 2 \Rightarrow k = 1$ (k is an integer)

(D) $\sin\theta = \cos\phi \Rightarrow \cos\left(\frac{\pi}{2} - \theta\right) = \cos\phi$

$\Rightarrow \frac{\pi}{2} - \theta = 2n\pi \pm \phi, n \in Z$

$\Rightarrow \theta \pm \phi - \frac{\pi}{2} = - 2n\pi, \pm n \in Z$

$\Rightarrow \frac{1}{\pi}\left(\theta \pm \phi - \frac{\pi}{2}\right) = - 2n, n \in Z$

Q.22 Consider all possible permutations of the letters of the word ENDEANOEL. Match the Statements/ Expressions in **column I** with the Statements/Expressions in **Column II** and indicate your answer by darkening the appropriate bubbles in the 4 x 4 matrix given in the ORS.

Column I	**Column II**
(A) The number of permutations containing the word ENDEA is	(p) 5!
(B) The number of permutations in which the Letter E occurs in the first and the last positions is	(q) 2 x 5!
(C) The number of permutations in which none of the letters D,L,N occurs in the last five positions is	(r) 7 x 5!
(D) The number of permutations in which the letters A,E,O occurs only in odd positions is	(s) 21 x 5!

Sol. A - p ; B - s ; C - q, D - q

(A) If ENDEA is fixed word then assume this as a single letter

Total letters = 5, total number of arrangement = 5!

(B) If E is at first and last places then

Total permutation = $\frac{7!}{2!}$ = 21 x 5!

(C) If D, L, N is not in last five position

$\leftarrow$ D,L,N,M $\rightarrow$ $\leftarrow$ E,E,E,A,O $\rightarrow$

Total permutation = $\frac{4!}{2!}$ x $\frac{5!}{3!}$ = 2 x 5!

(D) Total odd position = 5

Permutation of AEEEO are $\frac{5!}{3!}$

Total even position = 4

Permutation of N,N,D,L = $\frac{4!}{2!}$

Hence total permutation = $\frac{5!}{3!}$ x $\frac{4!}{2!}$ = 2 x 5!